

- (b) Show by induction that if A is (nonstrictly) *diagonally dominant*, that is, if each $b_j \neq 0$ and $|b_j| \geq |a_{j-1}| + |c_j|$ for $j = 1, \dots, n$, (with obvious modifications for $j = 1, n$), then all the $|u_j| \leq 1$. Show that if any $|u_j| < 1$, then all subsequent $|u_j| < 1$. Show that these conditions guarantee that Gaussian elimination never causes a divide by zero. Hence “pivoting” is not necessary.
- (c) Write a Matlab code to solve a general tridiagonal system. Show that it works by generating random vectors a, b, c , and x (you can neglect diagonal dominance), computing $y = Ax$, and inverting.
3. Use your tridiagonal solver to write an implicit code (any $0 \leq \theta \leq 1$) for $u_t = Du_{xx}$ on $X_L < x < X_R$, that can accept either Dirichlet or homogeneous Neumann conditions at $x = X_L, X_R$.

Pick some simple Fourier initial data ($\sin \kappa x, \cos \kappa x$, etc.) and compare your computed solution to the exact solution at a fixed time. Run your code in each of the following three scenarios

- (a) $\theta = 0$: $k = \lambda h^2$
 (b) $\theta = 1/2$: $k = \mu h$
 (c) $\theta = 1$: $k = \mu h$

Here h = space step, k = time step, and λ, μ are fixed as $h, k \rightarrow 0$. These relationships come from stability analysis as discussed in class.

The result will be a set of six graphs (which can be superimposed): error E as a function of h and of k in each of the three scenarios. In a log-log plot, you should see straight lines; that is, $E \sim Ch^p$ and $E \sim Ck^q$ with six exponents (slopes) $p_0, p_{1/2}, p_1, q_0, q_{1/2}, q_1$. Explain the observed slopes in terms of truncation error analysis.

Also, keep track of the runtimes for $\theta = 0, \theta = 1/2$, and $\theta = 1$. For each value of θ , find the amount of time it would take to reach an error of 10^{-10} . To fit the exponents, have a look at how I implemented the least-squares approximation in class on Jan 29.

4. Consider $u_t = u_{xx}$ on $-\infty < x < \infty, t > 0$, with the discontinuous step function initial data

$$u(x, 0) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

Standard convergence theory does not apply with non-smooth initial data (why not?), so we have to explore the errors experimentally. Also,

the problem is given on an infinite domain, so you will need to assess whether the finite boundaries are affecting your solution.

- (a) What is the exact solution? (Hint: look for $u(x, t) = U(x/\sqrt{t})$ and find an ODE satisfied by $U(\xi)$.) Is this solution ever negative?
- (b) Solve this using your code from the previous problem. I suggest using homogeneous Neumann boundary conditions, on a symmetric domain $X_L = -X_R$. Is the numerical solution always ≥ 0 ? (Hint: look at $\theta = 0$ with k near its maximum stable value.)
- (c) Explore several different domain sizes, that is, different values of $X_L = -X_R$. At about what time do the boundaries start to influence the solution, and how does this time depend on X_R ?