

## Mat1062: Computational Methods for PDE

### Problem Set 1

Thursday January 17, 2008

due: Tuesday January 29

To save figures you can use the command "print -dps figure1.ps".

1. Given the heat equation  $u_t = u_{xx}$  on  $[0, L]$  with linear homogeneous boundary conditions, we seek exact solutions of the form

$$u(x, t) = T(t)X(x)$$

Plugging this assumption into the PDE, one finds

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = k.$$

for some real number  $k$ . This implies that  $T(t)$  and  $X(x)$  satisfy the ODEs

$$T'(t) = k\alpha^2 T(t), \quad X''(x) = kX(x).$$

If the constant  $k$  is a negative number then one can write  $k$  as  $-\lambda^2$  and the ODEs become

$$T'(t) = -\lambda^2 \alpha^2 T(t), \quad X''(x) = -\lambda^2 X(x),$$

resulting in the solution  $X(x) = A \cos(\lambda x) + B \sin(\lambda x)$  for constants  $A$  and  $B$  and a  $T(t)$  that decays to zero as  $t \rightarrow \infty$ .

If the constant  $k$  is zero then the ODEs become

$$T'(t) = 0, \quad X''(x) = 0,$$

resulting in the solution  $X(x) = Ax + B$  for constants  $A$  and  $B$  and a  $T(t)$  that is constant.

If the constant  $k$  is a positive number then one can write  $k$  as  $\lambda^2$  and the ODEs become

$$T'(t) = \lambda^2 \alpha^2 T(t), \quad X''(x) = \lambda^2 X(x),$$

resulting in the solution  $X(x) = A \cosh(\lambda x) + B \sinh(\lambda x)$  for constants  $A$  and  $B$  and a  $T(t)$  that goes to infinity as  $t \rightarrow \infty$ .

- (a) If the IBVP has Dirichlet boundary conditions

$$u(0, t) = 0, \quad u(L, t) = 0$$

prove that there are no nontrivial solutions  $X(x)$  when  $k \geq 0$ . That is, all separable solutions decay in time.

- (b) If the IBVP has the Neumann boundary conditions

$$u_x(0, t) = 0, \quad u_x(L, t) = 0$$

prove that there are no nontrivial solutions  $X(x)$  when  $k > 0$ .

- (c) If the IBVP has Robin boundary conditions

$$-u_x(0, t) = -\frac{h_1}{k}u(0, t), \quad u_x(L, t) = -\frac{h_2}{k}u(L, t)$$

where  $h_1$ ,  $h_2$ , and  $k$  are all positive numbers, show that there are no nontrivial solutions  $X(x)$  when  $k > 0$ . What about if  $k = 0$ ?

- (d) If the IBVP has Robin boundary conditions

$$u_x(0, t) + 3u(0, t) = 0, \quad u_x(2, t) - 8u(2, t) = 0$$

There are two nontrivial  $X(x)$  which correspond to  $k > 0$  solutions. Find them. (You will need to use a computer to help you to do this.)

2. We now want to ask the same questions as above except for the discrete problem.

- (a) Write a matlab function called `DirichletMatrix.m` which has two inputs: the length of the interval  $[x_L, x_R]$  and the number of subintervals. It has one output: the matrix  $M_2$  as on page 3 of the lecture notes.
- (b) Take  $N = 8$ ,  $L = 1$  and call the output of your function `DirichletMatrix`. Use matlab's "eig" command to find the eigenvalues of  $M_2$ . You can find the largest three eigenvalues via the commands:  $v = \text{sort}(\text{eig}(M_2)); v(N - 2 : N)$ . Analytically, you know what the true eigenvalues are for the PDE on the interval with Dirichlet boundary conditions. If  $V_2$  is the value for the second largest eigenvalue of the PDE problem, let  $\text{err}(1) = v(2) - V_2$  be the difference between it and the second largest eigenvalue of the matrix.
- (c) Repeat the above with  $N = 16$  and define  $\text{err}(2)$  analogously.
- (d) Repeat the above with  $N = 32$  and define  $\text{err}(3)$  analogously.
- (e) Repeat the above with  $N = 64$  and define  $\text{err}(4)$  analogously.
- (f) Repeat the above with  $N = 128$  and define  $\text{err}(5)$  analogously.
- (g) Repeat the above with  $N = 256$  and define  $\text{err}(6)$  analogously.
- (h) Give the six values of  $\text{err}$ . (Note: if you type "format short e" you'll get a better display of the error. You should notice they are decreasing. Now give the subsequent ratios  $\text{err}(1)/\text{err}(2)$  and so on. What do you see? (Note: if you type "err(1:5)./err(2:6)" you'll get all those ratios at once.)
- (i) What are the eigenvectors of the matrix  $M_2$ ?

3. We want to solve the partial differential equation for  $u(x, t)$

$$u_t = D u_{xx} \quad \text{for } 0 \leq x \leq L \text{ and } t \geq 0,$$

with initial and boundary data (we suppose  $f(0) = f(L) = 0$ )

$$u(x, 0) = f(x) \quad \text{for } 0 \leq x \leq L; \quad u(0, t) = u(L, t) = 0 \quad \text{for } t \geq 0.$$

- (a) Write a Matlab program to compute  $u_j^n$  for  $n = 0, 1, \dots$ . How you visualize the solution is up to you. If you are motivated and have time, an excellent way to see what is going on is to use the `mesh` command to plot the surface  $u(x, t)$ . This is, however, no substitute for making quantitative comparisons as in the next item.
- (b) For  $f(x) = \sin k\pi x/L$ , with  $k$  an integer, you can easily compute the exact solution to this PDE. (What is it?) Take  $D = 1$ ; take  $N = 10, 20, 50, 100, \dots$ , and take  $k = \frac{1}{4}h^2$ . Measure the maximum difference between the solution of your difference formula at  $t = 1$  and the exact solution (maximum over  $x$  at fixed  $t$ ). Calling the maximum  $e_N$ , make a log-log plot of  $e_N$  vs.  $N$ , and show that  $e_N \rightarrow 0$  as  $N \rightarrow \infty$ .
- (c) Consider the initial data (suppose  $N$  is even)

$$u_j^0 = \begin{cases} 1/h, & j = N/2 \\ 0, & \text{else.} \end{cases}$$

What “function”  $f(x)$  does this approximate? What will the solution look like for short times? For long times? Run your code for this initial data and compare the solution to the approximate solution of the PDE computed as though the spatial domain were infinite.

As a measure of how much the boundary conditions are affecting your solution, plot the quantity

$$I^n = h \left( \frac{1}{2}(u_0 + u_N) + \sum_{j=1}^{N-1} u_j^n \right) \approx I(t) = \int_0^L u(x, t) dx$$

as a function of time. How does this behave for short times, and why does it help you distinguish between short and long times?

- (d) Make a copy of your code and modify it to take *homogeneous Neumann* boundary conditions

$$u_x(0, t) = u_x(L, t) = 0, \quad t \geq 0$$

(assume now that the initial data  $f(x)$  satisfies this condition).

Repeat a), b), and c) above, taking  $f(x) = \cos k\pi x/L$ . How does the profile of  $I(t)$  change? (How does  $I(t)$  evolve in your discrete model?)