Mat1062: Introductory Numerical Methods for PDE

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1 Ownership

These notes are the joint property of Rob Almgren and Mary Pugh.

2 Convergence of the Approximate Solutions

We recall the Hilbert space

$$V = \{v(x) \text{ real-valued functions on } \mathbb{R} \\ \left| \int_0^1 v(x)^2 \, dx < \infty, \int_0^1 v'(x)^2 \, dx < \infty, v(0) = 0 \right\} \quad (1)$$

and the inner products

$$a(u, v) = \int_0^1 u'(x) \, v'(x) \, dx \qquad \langle f, v \rangle = \int_0^1 f(x) \, v(x) \, dx.$$

Assume $f \in L^2([0, 1])$ then

$$u \text{ is a weak solution} \iff a(u, v) = \langle f, v \rangle \quad \text{for all } v \in V$$

Similarly, given an n-dimensional subspace $V_n \subset V$

$$u_n \text{ solves the Ritz-Galerkin approximation} \\ \iff a(u, v_n) = \langle f, v_n \rangle \quad \text{for all } v_n \in V_n$$

We now turn to the question: does u_n converge to a weak solution u as $n \to \infty$?

2.1 Projections

We would like to know whether u_n is the projection of u onto the subspace V_n. That is, of all the elements of V_n is u_n the one that's closest to u? The answer turns out to be "yes" if we use the right inner product/norm. But before we prove this, we need some preliminaries.

We have two inner products on V: $a(\cdot,\cdot)$ and $\langle \cdot,\cdot \rangle$. These inner products induce two norms:

$$\|u\| := \sqrt{\int_0^1 u^2} = \sqrt{\langle u, u\rangle} \qquad \|u\|_E := \sqrt{\int_0^1 u'^2} = \sqrt{a(u,u)}$$

We know $a(u_n, v_n) = \langle f, v_n\rangle$ for all $v_n \in V_n$. Similarly, we know $a(u,v) = \langle f, v\rangle$ for all $v \in V$. Since $V_n \subset V$ this implies $a(u, v_n) = \langle f, v_n\rangle$ for all $v_n \in V_n$. Therefore

$$a(u_n, v_n) = \langle f, v_n\rangle = a(u, v_n) \quad \forall\, v_n \in V_n \quad \Longrightarrow \quad a(u-u_n, v_n) = 0 \quad \forall\, v_n \in V_n$$

This shows that $u - u_n$ is perpendicular to the subspace V_n with respect to the $a(\cdot,\cdot)$ inner product. Given this observation, the following theorem is very natural.

Projection Theorem *Assume* $f \in L^2([0,1])$, V_n *is a subspace of* V, u_n *is a solution of the Ritz-Galerkin approximation problem and* u *is a weak solution. Then*

$$\|u - u_n\|_E = \inf_{v_n \in V_n} \|u - v_n\|_E.$$

Proof First of all, because $u_n \in V_n$ it's automatically true that

$$\inf_{v_n \in V_n} \|u - v_n\|_E \le \|u - u_n\|_E.$$

We now prove the opposite inequality, which will then imply that the two sides are equal.

$$\begin{aligned}\|u - u_n\|_E^2 &= a(u-u_n, u-u_n) = a(u-u_n, u-v_n) + a(u-u_n, v_n-u_n) \\ &= a(u-u_n, u-v_n) \quad \forall\, v_n \in V_n.\end{aligned}$$

In the last step, we used that $v_n - u_n \in V_n$ and that $u - u_n$ is perpendicular to V_n with respect to the $a(\cdot,\cdot)$ inner product. Applying the Schwartz inequality,

$$\|u - u_n\|_E^2 = a(u - u_n, u - v_n) \leq \|u - u_n\|_E \, \|u - v_n\|_E \quad \forall\, v_n \in V_n$$

Hence

$$\|u - u_n\|_E \leq \|u - v_n\|_E \quad \forall\, v_n \in V_n$$

Therefore

$$\|u - u_n\|_E \leq \inf_{v_n \in V_n} \|u - v_n\|_E,$$

finishing the proof.

Now we would like to bound $\|u - u_n\|_E$ with something that doesn't depend on u. We can do this under certain circumstances. For example, assume that our vector spaces satisfy:

Approximation Assumption *There is a constant* ϵ *such that for any* $w \in V$

$$\int_0^1 w_{xx}^2(x)\, dx < \infty \qquad \Longrightarrow \qquad \inf_{v_n \in V_n} \|w - v_n\|_E \leq \epsilon \|w''\| \tag{2}$$

Not all elements w of V will have $\int w_{xx}^2 < \infty$. The approximation assumption only applies to those vectors that have a second derivative and whose second derivative is square-integrable. If V and V_n are such that this approximation assumption holds then we can bound $\|u - u_n\|_E$ with something that doesn't depend on u. Specifically, we can prove an upper bound that depends on the data f:

Theorem *If* $f \in L^2([0,1])$ *and* u *is a weak solution and* u_n *is a solution of the Ritz-Galerkin approximation problem and if the approximation assumption (2) holds then*

$$\|u - u_n\| \leq \epsilon \|u - u_n\|_E \leq \epsilon^2 \|f\| \tag{3}$$

Proof: We use a duality argument to prove the first inequality. Specifically, we introduce a different, but related, PDE. Let w be a weak solution of

$$\begin{cases} -w'' = u - u_n & \text{on } (0,1) \\ w(0) = 0 \\ w'(1) = 0 \end{cases}$$

Then

$$\begin{aligned} \|u - u_n\|^2 &= \int_0^1 (u(x) - u_n(x))(u(x) - u_n(x))\, dx \\ &= -\int_0^1 (u(x) - u_n(x)) w''(x)\, dx = \int_0^1 (u'(x) - u_n'(x)) w'(x)\, dx \\ &= a(u - u_n, w) = a(u - u_n, w - v_n) \quad \forall\, v_n \in V_n \end{aligned}$$

In the last step, we used that $a(u - u_n, v_n) = 0$ for all $v_n \in V_n$. Therefore

$$\|u - u_n\|^2 \leq \|u - u_n\|_E \|w - v_n\|_E \qquad \forall\, v_n \in V_n$$

hence

$$\|u - u_n\|^2 \leq \|u - u_n\|_E \inf_{v_n \in V_n} \|w - v_n\|_E$$

We would like to apply the approximation assumption (2). We know that $u - u_n \in L^2$ and $w'' = u - u_n$[1]. Hence $w'' \in L^2$ and we can use the approximation assumption. It then follows that

$$\|u - u_n\|^2 \leq \|u - u_n\|_E \epsilon \|w''\| = \epsilon \|u - u_n\|_E \|u - u_n\|$$

Dividing both sides of the inequality by $\|u - u_n\|$ yields the first inequality in (3).

$$\|u - u_n\| \leq \epsilon \|u - u_n\|_E.$$

It remains to show that

$$\|u - u_n\|_E \leq \epsilon \|f\|.$$

We know by the projection theorem that

$$\|u - u_n\|_E = \inf_{v_n \in V_n} \|u - v_n\|_E$$

[1] See caveat at end of proof.

Because $f \in L^2$ and $u'' = f$[2] we know that $u'' \in L^2$ and so the approximation assumption (2) applies yielding

$$\inf_{v_n \in V_n} \|u - v_n\|_E \leq \epsilon \|u''\| = \epsilon \|f\|$$

proving the desired second inequality. This finishes the proof.

In the above proof, I was a little quick on a few details. For example, just because $f \in L^2$ it's not obvious that $u'' \in L^2$ unless we happen to know that u is actually a classical solution. If we knew that u were a classical solution then we'd know that $u'' = f$ pointwise and therefore $u \in L^2$. (This is what I used in the proof.) In fact, this is actually kosher for many elliptic PDE — a common approach for many elliptic PDE problems is: 1) Existence: prove that there is a weak solution using functional analysis methods, 2) Regularity: prove that the weak solution has enough derivatives to be a classical solution. I'm not going to go into this because this isn't a course in PDE.

2.2 Interpolants

Now that we know

$$\|u - u_n\| \leq \epsilon^2 \|f\|$$

for spaces that satisfy the approximation assumption (2) we'd love to know some spaces that do satisfy the assumption and, better yet, we'd like to know that $\epsilon \to 0$ as $n \to \infty$.

We recall the n-dimensional space of piecewise linear functions. Fix a set of $n + 1$ points in $[0, 1]$ such that

$$0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1.$$

We call these points "nodes". Using these $n+1$ nodes, we create n piecewise linear functions ϕ_j such that $\phi_j(x_j) = 1$ for all $1 \leq j \leq n$ and such that the support of ϕ_j is $[x_{j-1}, x_{j+1}]$.

[2] See caveat at end of proof.

Definition *Given* v *continuous on* $[0,1]$*, the interpolant of* v *is in* V_n *and is defined by*

$$v_I(x) := \sum_{j=1}^{n} v(x_j)\ \phi_j(x)$$

In other words, given v *and a collection of nodes,* v_I *is what you get if you "connect-the-dots" on the graph using the node points* $(x_j, v(x_j))$*.*

We start by proving what we observed in the computation: u_n and u agree at the nodes. To show this, it suffices to show that $u_n = u_I$.

Theorem:*If* $f \in L^2([0,1])$ *and* u *is a weak solution and* u_n *is a solution of the Ritz-Galerkin approximation problem and* u_I *is the interpolant of* u *in* V_n *then* $u_n = u_I$*.*

Proof: We prove this by showing that

$$a(u_I - u_n, v_n) = 0, \qquad \forall\, v_n \in V_n \tag{4}$$

Because $u_I - u_n \in V_n$ this will then imply $a(u_I - u_n, u_I - u_n) = 0$ which then implies $u_I - u_n = 0$, as desired.

To show (4) we prove

$$a(u - u_n, v_n) = a(u_I - u_n, v_n) \qquad \forall\, v_n \in V_n.$$

This would suffice because we already know the left-hand side equals zero. And so we want to show

$$\int_0^1 (u - u_n)'\, v_n'\, dx = \int_0^1 (u_I - u_n)'\, v_n'\, dx \qquad \forall\, v_n \in V_n.$$

It suffices to show that on each interval $[x_{i-1}, x_i]$

$$\int_{x_{i-1}}^{x_i} (u - u_n)'\, v_n'\, dx = \int_{x_{i-1}}^{x_i} (u_I - u_n)'\, v_n'\, dx \qquad \forall\, v_n \in V_n.$$

Fix $v_n \in V_n$ and fix i. We know that v_n is piecewise linear on $[0,1]$ and is linear on $[x_{i-1}, x_i]$. (The corners in the graph of v_n can only occur at nodes.) As a result, we know that $v_n' = m$ on $[x_{i-1}, x_i]$ for some number m. And so we want to show

$$\int_{x_{i-1}}^{x_i} (u - u_n)'\ m\ dx = \int_{x_{i-1}}^{x_i} (u_I - u_n)'\ m\ dx$$

We're done by the Fundamental Theorem of Caclulus:

$$\int_{x_{i-1}}^{x_i} (u - u_n)' \, m \, dx = m \, [u(x_i) - u_n(x_i) - u(x_{i-1}) + u_n(x_{i-1})]$$
$$= m \, [u_I(x_i) - u_n(x_i) - u_I(x_{i-1}) + u_n(x_{i-1})] = \int_{x_{i-1}}^{x_i} (u_I - u_n)' \, m \, dx$$

Above, I used that u and u_I agree at the nodes. This finishes the proof.

We now show that the approximation assumption (2) holds for this space. Because (2) involves an infimum taken over all piecewise linear functions v it suffices to show that the inequality holds for a particular piecewise linear function. Specifically, we'll show the inequality holds if we take the linear interpolant, w_I, as a sample piecewise linear function.

Theorem: *If* $w'' \in L^2$ *and* $h = \max\{x_j - x_{j-1}\}$ *then*

$$\|w - w_I\|_E \leq \frac{h}{\sqrt{2}} \|w''\|$$

Therefore the approximation assumption (2) holds with $\epsilon = h/\sqrt{2}$.

Proof: Since

$$\|w - w_I\|_E^2 = \int_0^1 (w'(x) - w_I'(x))^2 \, dx = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} (w'(x) - w_I'(x))^2 \, dx,$$

it suffices to prove the desired bound on each subinterval. And so we will prove

$$\int_{x_{j-1}}^{x_j} (w'(x) - w_I'(x))^2 \, dx \leq \tfrac{1}{2} \, (x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} w''(x)^2 \, dx \qquad \forall \, 1 \leq j \leq n$$

First, we change variables, introducing

$$\widetilde{x} = \frac{x - x_{j-1}}{x_j - x_{j-1}} \quad \Longrightarrow \quad \frac{d}{dx} = \frac{1}{x_j - x_{j-1}} \frac{d}{d\widetilde{x}} \quad \text{and} \quad (x_j - x_{j-1}) \, d\widetilde{x} = dx$$

In the new variables, we see it suffices to prove

$$\int_0^1 (w'(\widetilde{x}) - w_I'(\widetilde{x}))^2 \, d\widetilde{x} \leq \tfrac{1}{2} \int_0^1 w''(\widetilde{x})^2 \, d\widetilde{x}.$$

In the new coordinates, $w - w_I$ vanishes at both $x = 0$ and $x = 1$. Because $\int w''^2 < \infty$, we know that w' is continuous on $[0, 1]$. We also know that w_I' is continuous on $[0, 1]$ and therefore $w' - w_I'$ is too. And so $w - w_I$ is a continuous function which vanishes at $x = 0$ and $x = 1$. Therefore, by the mean value theorem, there is some point ξ at which $w'(\xi) - w_I'(\xi) = 0$. Hence

$$w'(y) - w_I'(y) = \int_\xi^y w''(x) - w_I''(x)\ dx = \int_\xi^y w''(x)\ dx.$$

Above, we used that since w_I is linear, $w_I'' = 0$. By the Schwartz inequality,

$$\begin{aligned} \left|w'(y) - w_I'(y)\right| &= \left|\int_\xi^y w''(x)\ dx\right| \leq \sqrt{\int_\xi^y 1\ dx}\sqrt{\int_\xi^y w''(x)^2\ dx} \\ &= \sqrt{|\xi - y|}\sqrt{\int_\xi^y w''(x)^2\ dx} \leq \sqrt{|\xi - y|}\sqrt{\int_0^1 w''(x)^2\ dx} \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^1 \left|w'(y) - w_I'(y)\right|^2\ dy &\leq \int_0^1 |\xi - y|\ \int_0^1 w''(x)^2\ dx\ dy \\ &= \int_0^1 w''(x)^2\ dx\ \int_0^1 |\xi - y|\ dy \leq \tfrac{1}{2}\int_0^1 w''(x)^2\ dx \end{aligned}$$

In the last step, we used that

$$\sup_{0<\xi<1} \int_0^1 |\xi - y|\ dy = \tfrac{1}{2}.$$

This finishes the proof.

Note: I could have had a cruder upper bound in the last step and used that $|\xi - y| \leq 1$ implies that $\int_0^1 |\xi - y|\ dy \leq 1$. This would have resulted in the slightly larger value of $\epsilon = h$ rather than $\epsilon = h/\sqrt{2}$. This would have had no effect on the convergence result or on the rate of convergence. I kept the $\sqrt{2}$ around because it results in a sharp constant (you can't find a smaller value for ϵ) and it's a force of habit for me as a mathematician.

Again, in the above proof I was quick on a few details. I constructed $w' - w_I'$ from w'' using the fundamental theorem of calculus. This holds only if $w' - w_I'$ is absolutely continuous. However I can get the desired bound via density arguments and the Lebesgue Dominated Convergence theorem, in a similar manner as I did in the proof of the Lax-Wendroff theorem.

We therefore know that for the subspace of linear interpolants the inequality (3) becomes

$$\|u - u_n\| \leq \frac{1}{\sqrt{2}} \max\{x_j - x_{j-1}\} \, \|u - u_n\|_E \leq \tfrac{1}{2} \left(\max\{x_j - x_{j-1}\}\right)^2 \|f\|$$

This shows that if we choose the nodes so that the maximum distance between nodes goes to zero as $n \to \infty$ then $u_n \to u$ in both the L^2 norm and in the energy norm.

3 How to compute $\langle f, \phi_j \rangle$

To find u_n we need to compute

$$\langle f, \phi_j \rangle = \int_0^1 f(x) \, \phi_j(x) \, dx = \int_{x_{j-1}}^{x_{j+1}} f(x) \, \phi_j(x) \, dx \qquad \forall \, j.$$

Rather than using a quadrature method in which we divide the interval $[x_{j-1}, x_{j+1}]$ into m subintervals and approximate the integral on each subinterval, we use Gaussian Quadrature.

Gaussian Quadrature approximates an integral by sampling the integrand at certain very well-chosen nodes and then making a weighted sum out of those samples:

$$\int_{-1}^1 g(x) \, dx \approx \sum_{i=1}^N w_i \, g(x_i)$$

For example, if we use 5 sample points then the weights and sample points are

x_i	w_i
0	$128/225$
$\pm\sqrt{5 - 2\sqrt{10/7}}$	$(322 + 13\sqrt{70})/900$
$\pm\sqrt{5 + 2\sqrt{10/7}}$	$(322 - 13\sqrt{70})/900$

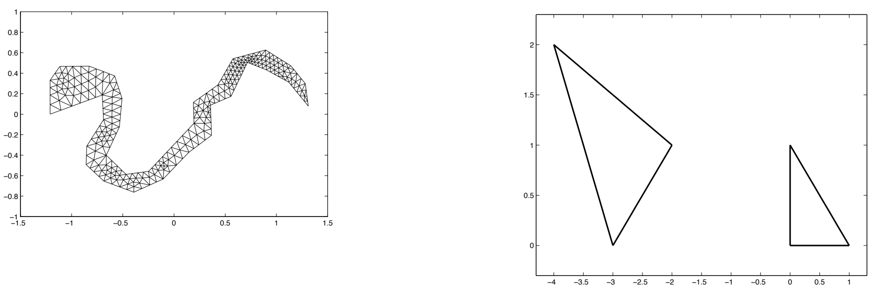

Figure 1: Left plot: a sample domain in the plane and triangularization. Right: the triangle on the left is an arbitrary triangle Δ. The triangle on the right is a reference triangle, Δ_0, for which good Gaussian Quadrature rules are known.

A simple change of variables allows us to use Gaussian Quadrature on an interval:

$$\int_a^b g(x)\,dx \approx \frac{b-a}{2}\sum_{i=1}^{N} w_i\, g\left(\frac{b-a}{2}\,x_i + \frac{a+b}{2}\right) \tag{5}$$

Given a domain in the plane, the first step is to construct a triangularization. See the left plot in Figure 1, for example. Similarly, given a domain in three-space, the first step is to decompose the domain into tetrahedrons. Basis functions are then defined and things proceed as in the one-dimensional example. To find

$$\int_\Delta f(x,y)\,\phi_j(x,y)\,dxdy$$

where Δ is an arbitrary triangle in the plane, we first change coordinates so that we need to compute

$$\int_{\Delta_0} f(\tilde{x},\tilde{y})\,\phi_j(\tilde{x},\tilde{y})\,d\tilde{x}d\tilde{y}$$

where Δ_0 is the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$. See Figure 1. To approximate the integral over Δ_0 we use a Gaussian Quadrature rule and sample at certain well-chosen points in Δ_0. This leads to a formula analogous to (5).