

Mat1062: Introductory Numerical Methods for PDE

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1 Ownership

These notes are the joint property of Rob Almgren and Mary Pugh.

2 The Ritz-Galerkin Approximation Problem

We seek approximate solutions of

$$\mathbf{u} \in V, \quad \mathbf{a}(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V \quad (1)$$

where

$V = \{v(x) \text{ real-valued functions on } \mathbb{R}$

$$\left. \left| \int_0^1 v(x)^2 dx < \infty, \int_0^1 v'(x)^2 dx < \infty, v(0) = 0 \right\}$$

We do this via subspaces of V . That is, we will consider a family of subspaces $\cdots \subset V_n \subset V_{n+1} \subset \cdots \subset V$ and in each subspace we will find and solve an approximate problem, resulting in a solution \mathbf{u}_n . The goal is to choose the subspaces in a smart way, one that allows us to show that \mathbf{u}_n converges to something, call it \mathbf{u} , as $n \rightarrow \infty$ and that the limit \mathbf{u} is a weak solution in the sense (1).

Given a subspace $V_n \subset V$, The Ritz-Galerkin approximation problem is

$$\mathbf{u}_n \in V_n, \quad \mathbf{a}(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V_n \quad (2)$$

If V_n is a finite-dimensional subspace then there will always be a solution \mathbf{u}_n and it will be unique:

Theorem Assume V_n is an n -dimensional subspace of V and $f \in L^2([0, 1])$. Then the Ritz-Galerkin approximation problem (2) has a unique solution u_n .

Proof: Let $\{\phi_j\}$ be a basis of V_n . First of all, if $v \in V_n$ then

$$v = \sum_{j=1}^n V_j \phi_j \implies a(u_n, v) = \langle f, v \rangle \iff \sum_{j=1}^n V_j a(u_n, \phi_j) = \sum_{j=1}^n V_j \langle f, \phi_j \rangle$$

as a result, it suffices to find $u_n \in V_n$ such that

$$a(u_n, \phi_j) = \langle f, \phi_j \rangle, \quad \forall 1 \leq j \leq n.$$

We seek U_1, U_2, \dots, U_n such that

$$u_n = \sum_{i=1}^n U_i \phi_i \implies a(u_n, \phi_j) = \langle f, \phi_j \rangle \iff \sum_{i=1}^n U_i a(\phi_i, \phi_j) = \langle f, \phi_j \rangle \quad \forall 1 \leq j \leq n.$$

This is a linear algebra problem. If

$$\vec{U} = \begin{pmatrix} U_1 \\ \dots \\ U_n \end{pmatrix}, \quad \vec{F} = \begin{pmatrix} \langle f, \phi_1 \rangle \\ \dots \\ \langle f, \phi_n \rangle \end{pmatrix}, \quad K_{ij} = a(\phi_i, \phi_j)$$

then we seek a solution \vec{U} of

$$\vec{U}^T K = \vec{F}^T$$

There will be a unique solution if and only if the null space contains only the zero vector. That is, we need to show

$$\vec{V}^T K = \vec{0}^T \iff \vec{V} = \vec{0}.$$

We see this as follows:

$$\begin{aligned} \vec{V}^T K = \vec{0}^T &\implies \langle \vec{V}^T K, \vec{V} \rangle = 0 \\ &\implies a(v, v) = 0 \quad \text{where } v = \sum V_j \phi_j \\ &\implies v(x) = 0 \quad \forall x \in [0, 1] \\ &\implies \vec{V} = \vec{0}. \end{aligned}$$

In the last step, we used that $\{\phi_j\}$ is a basis and therefore the only way a linear combination of ϕ_j s can equal zero is if each coefficient equals zero.

This shows that the null space is trivial which implies there exists a unique solution u_n , as desired. This finishes the proof.

The proof is helpful in that it shows us how to solve for u_n once we have a basis for the subspace V_n .

2.1 A piecewise linear basis

We now consider a collection of functions which we hope can become a basis for V . Fix a set of $n + 1$ points in $[0, 1]$ such that

$$0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1.$$

We call these points “nodes”. Using these nodes, we create n piecewise linear functions ϕ_j . First, let $dx_j := x_j - x_{j-1}$ for $1 \leq j \leq n$. Then for $1 \leq j \leq n - 1$

$$\phi_j(x) = \begin{cases} \frac{1}{dx_j} (x - x_{j-1}) & x_{j-1} \leq x \leq x_j \\ \frac{-1}{dx_{j+1}} (x - x_{j+1}) & x_j < x \leq x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\phi_n(x) = \begin{cases} \frac{1}{dx_n} (x - x_{n-1}) & x_{n-1} \leq x \leq x_n \\ 0 & \text{otherwise} \end{cases}.$$

Each ϕ_j is 1 at $x = x_j$. Also, note that each ϕ_j is zero at $x = 0$ and that ϕ_n is nonzero at $x = 1$. Clearly, if a linear combination of the ϕ_j s is zero then each coefficient must be zero, proving linear independence.

$$\begin{aligned} \sum_{i=1}^n C_i \phi_i(x) &= 0 \quad \forall x \in [0, 1] \\ \implies \sum_{i=1}^n C_i \phi_i(x_j) &= C_j \phi_j(x_j) = 0 \quad \forall 1 \leq j \leq n \\ \implies C_j &= 0 \quad \forall 1 \leq j \leq n \end{aligned}$$

We'd like to show that as we add more and more nodes that the set expands to become (in the limit) a basis for V . We will address this later.

We take $V_n := \text{span}\{\phi_j\}$. Rather than starting with a subspace and finding a basis for it, we start with a set of linearly independent functions and take their span to be the subspace V_n .

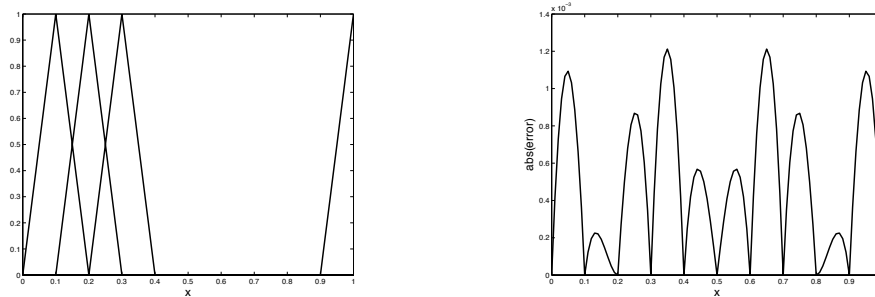


Figure 1: The eleven nodes are uniformly spaced in $[0, 1]$ with $dx = 1/10$. Left plot: the first three basis functions, ϕ_1, ϕ_2, ϕ_3 are plotted along with the last basis function, ϕ_{10} . Right plot: We take $f(x) = \cos(3\pi x)$. We solve the Ritz-Galerkin approximation problem and construct u_{10} . Plotted here is the absolute value of the error, sampled at 101 points.

To solve the Ritz-Galerkin approximation problem, we need the matrix K and the vector \vec{F} . Because the elements ϕ_j are piecewise linear, their derivatives are piecewise constant which makes it easy to compute $K_{i,j} = a(\phi_i, \phi_j)$. For $1 \leq i < n$

$$\begin{cases} K_{i,i-1} = -\frac{1}{dx_i} \\ K_{i,i} = \frac{1}{dx_i} + \frac{1}{dx_{i+1}} \\ K_{i,i+1} = -\frac{1}{dx_{i+1}} \end{cases}$$

and

$$\begin{cases} K_{n,n-1} = -\frac{1}{dx_n} \\ K_{n,n} = \frac{1}{dx_n} \end{cases}$$

2.2 Let's compute!

I take $f(x) = \cos(3\pi x)$, yielding the exact solution

$$u(x) = \frac{1}{9\pi^2} \cos(3\pi x) - \frac{1}{9\pi^2}.$$

I use maple to compute $\langle f, \phi_j \rangle$. Maple can do this — it's just integration by parts. This produces a long, nasty formula that I then cut and paste into

my matlab program. I take the nodes to be uniformly spaced ($dx_j = 1/n$) resulting in

$$\begin{aligned} \langle f, \phi_j \rangle &= -\frac{2}{9dx\pi^2} \cos(\pi j dx) (\cos(\pi dx) - 1) \\ &\quad \cdot (1 + 2 \cos(\pi dx))^2 (-3 + 4 \cos(\pi j dx)^2) \quad \forall 1 \leq j \leq n-1 \\ \langle f, \phi_n \rangle &= \frac{1}{9\pi^2 dx} (\cos(\pi dx) - 1) (1 + 2 \cos(\pi dx))^2 \end{aligned}$$

I solve the system $\vec{U}^T K = \vec{F}^T$ and use \vec{U} to construct

$$u_n(x) = \sum_{j=1}^n U_j \phi_j(x).$$

I find that if I evaluate the error $u - u_n$ at the nodes x_j then I get zero to machine precision. And so to evaluate the error elsewhere, I need to sample u_n and u away from the nodes. Between each node, I sample at 9 equally spaced points. In the right plot of Figure 1, I present the absolute value of the error as a function of x . Its largest value (the L^∞ norm) is approximately $1.2e-2$.

I then test the scheme for convergence by doing seven runs. The first run has 11 nodes with uniform spacing $dx = 1/10$. The left plot of Figure 1 shows some of the basis functions. The second run has 21 nodes with uniform spacing $dx = 1/20$ and so on. The spacing decreases by a factor of two in each subsequent run. To measure the error, I sample on a uniform mesh with meshwidth $dx/10$. I compute the L^∞ norm of the error as well as the L^2 error.

number of nodes	$\ \text{err}\ _{L^\infty}$	$\ \text{err}\ _{L^2}$	ratio of L^2 errors
11	1.2e-3	6.3e-4	3.9421
21	3.1e-4	1.6e-4	3.9855
41	7.8e-5	4.0e-5	3.9964
81	2.0e-5	1.0e-5	3.9991
161	4.9e-6	2.5e-6	3.9998
321	1.2e-6	6.3e-7	3.9999
641	3.1e-7	1.7e-7	

We see that the L^2 norm of the error is decreasing by powers of 4. In fact, so is the L^∞ norm.

I now consider a non-uniform distribution of nodes. I do this by parametrizing the interval $[0, 1]$ via $x(s)$ where $x(0) = 0$ and $x(1) = 1$ and $x'(s) > 0$ (nonconstant). Specifically, I take

$$x(s) = s + \left(\frac{1}{2\pi} - \frac{1}{100} \right) \sin(2\pi s) \quad (3)$$

and then sample uniformly in s to produce nodes x_j . I chose this function because x' varies by a factor of 30. I start with 11 nodes, chosen by taking

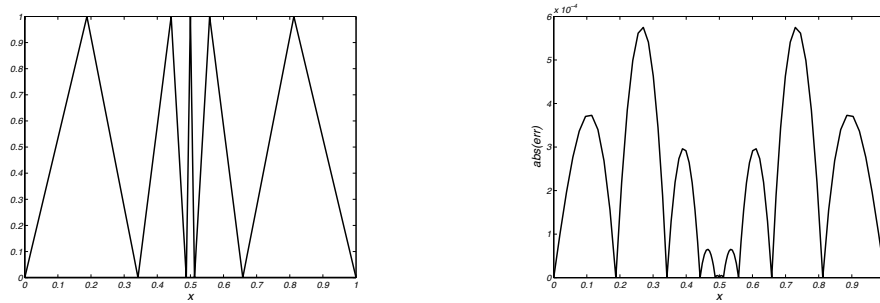


Figure 2: The eleven nodes are nonuniformly spaced in $[0, 1]$ by taking the parametrization (3) with $ds = 1/10$. Left plot: the odd-indexed basis functions, ϕ_1, \dots, ϕ_9 . Right plot: As before, we take $f(x) = \cos(3\pi x)$. We solve the Ritz-Galerkin approximation problem and construct u_{10} . Plotted here is the absolute value of the error, sampled at 101 points. Note that the error is smaller where the spatial resolution is finer.

11 equally spaced points s_j in $[0, 1]$ and applying the mapping (3). In the left plot of Figure 2, I show some of the basis functions for the case of 11 non-uniformly spaced nodes. The distance between nodes is greatest near $x = 0$ and $x = 1$ and is smallest near $x = 1/2$. In the right plot of Figure 2, I present the pointwise error. Note that the error is larger between the nodes that are further apart than between the nodes that are closer together. This is somewhat intuitive in that the function $f(x) = \cos(3\pi x)$ isn't especially different near $x = 1/2$ than near $x = 0$ and $x = 1$. You could imagine that if f had finer structure near $x = 1/2$ than near $x = 0$ and $x = 1$ then this might cause the errors near $x = 1/2$ to be comparable to, or larger than, the errors near $x = 0$ and $x = 1$.

Again, the errors decrease by a factor of 4 when I refine the mesh by factors of 2.

$\min(dx)$	$\max(dx)$	$\ err\ _{L^\infty}$	$\ err\ _{L^2}$	ratio of L^2 errors
1.2e-2	1.9e-1	2.7e-3	2.2e-3	3.6970
3.9e-3	9.6e-2	1.0e-3	5.8e-4	3.9402
1.7e-3	4.8e-2	2.8e-4	1.5e-4	3.9849
8.0e-4	2.4e-2	7.3e-5	3.7e-5	3.9962
3.9e-4	1.2e-2	1.8e-5	9.2e-6	3.9991
2.0e-4	6.1e-3	4.5e-6	2.3e-6	3.9998
9.8e-5	3.0e-3	1.1e-6	5.7e-7	