Marina Chugunova, notes from Thursday March 12

1. FINITE ELEMENT METHOD

Projections in Hilbert Space

Definition 1.1. *We say that a bilinear functional $a(u,v)$ defines **an inner product** in linear space B if it has the following properties (λ is an arbitrary complex number):*

1. $a(u,v) = \overline{a(v,u)}$
2. $a(u_1+u_2,v) = a(u_1,v)+a(u_2,v)$
3. $a(\lambda\, u,v) = \lambda\, a(u,v)$
4. $a(u,u) \geq 0$
5. $a(u,u) = 0 \Leftrightarrow u = 0$

(it follows from 3 (take $\lambda = 0$) that $a(0,v) = 0$.)

A Hilbert Space is a linear space that is complete with respect to the norm $||\cdot|| = \sqrt{(\cdot,\cdot)}$ induced by an inner product $(\cdot,\cdot)$.

(Not every norm can be induced by an inner product.)

Proposition 1.2. *A norm $||\cdot||$ is induced by an inner product $(\cdot,\cdot)$ if and only if the parallelogram equation*

$$||f_1+f_2||^2 + ||f_1-f_2||^2 = 2\,(||f_1||^2 + ||f_2||^2)$$

holds for any f_1, f_2.

Proof. This is a straight forward to show because

$$||f_1+f_2||^2+||f_1-f_2||^2 = (f_1+f_2,f_1+f_2)+(f_1-f_2,f_1-f_2) = 2\,(||f_1||^2+||f_2||^2)$$

□

Proposition 1.3. *If B_1 is a closed linear subspace of a Hilbert space B then B_1 is also a Hilbert space with respect to the same norm.*

Proof. It follows from the completeness of B that a Cauchy sequence in B_1 will converge to some element in B. This element will also belong to B_1 because B_1 is closed. Hence B_1 is complete. □

Example 1

Hilbert space:

$$L^2(0,1) := \quad \{f \in L^2(0,1),\ \int_0^1 |f(x)|^2 dx < \infty\}.$$

Inner product is given by:

$$(f,g) = \int_0^1 f(x)\overline{g(x)}dx$$

(For the proof of completeness see any functional analysis textbook) Check that the inner product satisfies properties $1-4$.

Property 5.

If $u(x) = 0$ then $(u, u) = \int_0^1 (0)^2 dx = 0$.

If $(u, u) = 0$? Compare two functions: $f_1(x) = 0, \quad x \in [0, 1]$ and $f_2(x) : \quad \{f_2(x) = 0, \quad x \in [0, 1/2) \cup (1/2, 0], \quad f_2(1/2) = 100\}$. We can see that $(f_1, f_1) = 0$ and $(f_2, f_2) = \int_0^1 |f_2|^2 dx = \int_0^{1/2} |f_2|^2 dx + \int_{1/2}^1 |f_2|^2 dx = 0$. But a linear space can have only one zero element. What is the zero element in $L^2(0, 1)$? This is not a function ! This is a class of functions which can differ from $f(x) = 0$ on the set of zero measure only (each class can have not more than one continuous function). This is the reason that "value at a point" notion is not applicable to elements from $L^2(0, 1)$ space.

Example 2

Sobolev space:

$$H^1(0,1): \quad \{f \in H^1(0,1), \int_0^1 |f(x)|^2 dx < \infty\}, \int_0^1 |f'(x)|^2 dx < \infty\}.$$

This is a Hilbert Space with respect to the inner product: $(f, g) = \int_0^1 f(x)\overline{g(x)}dx + \int_0^1 f'(x)\overline{g'(x)}dx$ where derivatives are understood in a weak sense to make the space complete (for example $f(x) = |x - 1/2|$ belongs to the space $H^1(0, 1)$).

Definition 1.4. *Let B_N be a subspace of B. We say that u_N is the projection of $u \in B$ to B_N if*

$$||u - u_N|| = \inf_{u_N^* \in B_N} ||u - u_N^*||.$$

(projection is the closest element from B_N to u)

Theorem 1.5. *If u_N is the projection of u to B_N then $u - u_N$ is orthogonal to $\forall\ h \in B_N$.* $\quad [\ (u - u_N, h) = 0, \quad \forall\ h \in B_N\ ]$

Proof. We will prove the statement by contradiction. Assume that exists such $h \neq 0 \in B_N$ that $(u - u_N, h) = \sigma \neq 0$. Construct a new element

$$h^* = u_N + \frac{\sigma}{(h,h)}\, h \in B_N$$

$$||u - h^*||^2 = (u - u_N - \frac{\sigma}{(h,h)}\, h, u - u_N - \frac{\sigma}{(h,h)}\, h) = ||u - u_N||^2 - \frac{\sigma^2}{(h,h)}$$

that means that h^* is closer to u than u_N and u_N is not the projection of u. □

Let B_N be a finite dimensional subspace of B. Let $\{\phi_1,\ \phi_2 ... \phi_N\}$ be a set of linearly independent functions in B_N, i.e $\{\phi_1,\ \phi_2 ... \phi_N\}$ forms a basis in B_N. That implies that for any element $f \in B_N$ exists the unique set of coefficients $(c_1, c_2 .... c_N)$ such that

$$f = c_1\, \phi_1 + c_2\, \phi_2 ... + c_N\, \phi_N.$$

Matrix

$$\Gamma = \begin{pmatrix} (\phi_1, \phi_1) & \cdots & (\phi_1, \phi_N) \\ \vdots & & \vdots \\ (\phi_N, \phi_1) & \cdots & (\phi_N, \phi_N) \end{pmatrix}$$

is called Gram Matrix.

Proposition 1.6. $\det(\Gamma) \neq 0$ *(Γ is invertible)*

Proof. Multiply $f = c_1\,\phi_1 + c_2\,\phi_2 ... + c_N\,\phi_N$ by $\phi_1,\ \phi_2\ ...\ \phi_N$ (by inner product). We obtain the linear system of equations:

$$\begin{cases} (\phi_1, f), & = c_1(\phi_1,\phi_1) + ... + c_N(\phi_1,\phi_N) \\ \vdots & \\ (\phi_N, f), & = c_1(\phi_N,\phi_1) + ... + c_N(\phi_N,\phi_N) \end{cases}$$

Because $\{\phi_1,\ \phi_2 ... \phi_N\}$ forms a basis the system above has the unique solution. Hence the $\det(\Gamma) \neq 0$. ☐

Projection of PDE

Let L be some closed linear operator acting in B. (For any closed linear operator the adjoint operator L^* is well defined.)

$$u_t = Lu + f, \quad u,\ f \in B, \quad L : B \to B \tag{1.1}$$

Let $u(x,t) \in C^2(0,1)$ be a classic solution of 1.1.

Let $B_N = span(\phi_1, \phi_2 ... \phi_N)$ (we have basis in B_N).

Let $\{\phi_1, \phi_2 ... \phi_N, \phi_{N+1} ...\}$ be some basis in B (the unique infinite series representation of any element in B converges strongly in the sense of the norm in B.)

Let u_N and f_N (see our definition) be the projections of u and f to B_N.

Under what conditions on $\phi_1, \phi_2 ... \phi_N$ does the equality

$$(u_{Nt}, \phi_n) = (L\,u_n, \phi_n) + (f_N, \phi_n), \quad n = 1..N$$

hold ?

By the Theorem 1.5:

$$(u - u_N, \phi_n) = 0, \quad (f - f_N, \phi_n), \quad n = 1..N.$$

$$(L(u - u_N), \phi_n) = (u - u_N, L^*\phi_n) = 0 \tag{1.2}$$

if $L^*\phi_n \in B_N, \quad n = 1..N$ (i.e if B_N is invariant under the adjoint operator L^*)

$$(u_t - u_{Nt}, \phi_n) = \frac{d}{dt}(u - u_N, \phi_n) = 0, \quad n = 1...N. \tag{1.3}$$

because ϕ_n does not depend on t.

It follows from the equation 1.1 that

$$(u_t, \phi_n) = (Lu, \phi_n) + (f, \phi_n), \quad n = 1..N. \tag{1.4}$$

From 1.2, 1.3, 1.4 it follows that

$$(u_{Nt}, \phi_n) = (L\,u_n, \phi_n) + (f_N, \phi_n), \quad n = 1..N.$$

u_N converges to u as $N \to \infty$ strongly by the norm in B because

$$\{\phi_1, \phi_2 ... \phi_N, \phi_{N+1} ...\}$$

forms a basis in B.

The solution u_N does make sense because this is the closest element from B_N to the exact solution u. (The solution of 1.4 exists and it is unique because Γ is invertible.)

Finite Element Method.

Consider the boundary value problem (BVP):

$$\begin{cases} -u'' = f, \forall\, x \in (0,1) \\ u(0) = 0 \\ u'(1) = 0 \end{cases}$$

Construct a suitable Hilbert Space (real valued functions):

$$V = \{v(x) \in V, \int_0^1 v^2 dx < \infty,\ \int_0^1 v'^2 dx < \infty,\ v(0) = 0\}$$

that is a subspace of the Sobolev Space $H^1(0,1)$ and as it follows from the proposition 1.3 V is a Hilbert space.

Definition 1.7. *We say that* $u(x) \in C^1(0,1), \quad u(0) = 0$ *is* ***a weak solution*** *of the* BVP *if*

$$\int_0^1 f\, v\, dx = \int_0^1 u'\, v'\, dx$$

for any $v(x) \in V$*.*

The definition of a weak solution comes from the integration by parts (assume that $u(x,t)$ is a classic solution of BVP: $u(0) = 0$ and $u'(1) = 0$ and $v(x) \in V, \quad v(0) = 0$):

$$\int_0^1 f\, v\, dx = -\int_0^1 u''\, v\, dx = \int_0^1 u'\, v' dx - u'(1)v(1) + u'(0)v(0) = \int_0^1 u'\, v' dx$$

Theorem 1.8. *Assume that* f *and* u'' *are continuous functions on* $[0,1]$*. If* $u \in V$ *and* u *is a weak solution of the* BVP *then* u *is a classical solution of the* BVP*.*

Proof. Given $u(0) = 0$ and $\int_0^1 f\, v\, dx = \int_0^1 u'\, v'\, dx$ for any $v(x) \in V$ we want to show that $u'(1) = 0$ and $-u'' = f$.

Integrating by parts:

$$\int_0^1 f\, v\, dx = \int_0^1 u'\, v'\, dx = -\int_0^1 u''\, v\, dx + u'(1)v(1)$$

We obtain:

$$u'(1)v(1) = \int_0^1 (f + u'')\, v\, dx \tag{1.5}$$

Define the new continuous function as $h(x) = f(x) + u''(x)$. Assume that $h(x)$ is not identically 0 on $[0,1]$. Then there exists a subinterval (δ_1, δ_2) of $(0,1)$ such that $h(x)$ is strictly positive (or strictly negative). In this case we just choose a test function $v(x)$ such that $v(x) \equiv 0$ on $(0,1)/(\delta_1, \delta_2)$ and

$v(x) > 0$ on (δ_1, δ_2). The equality 1.5 does not hold for this $v(x)$ hence $f(x) + u''(x) = 0$ and $u'(1) = 0$. □

Define a bilinear functional $a(u, v) = \int_0^1 u'\, v'\, dx$ and $(f, v) = \int_0^1 f(x)v(x)\, dx$.
Show that $a(u, v) = \int_0^1 u'\, v'\, dx$ is an inner product on V.
Check properties (1-4) by your own.
Property 5.

$$a(u, u) = 0 \quad \Leftrightarrow \quad u = 0$$

If $u = 0$ then $\int_0^1 (0)^2 dx = 0$
If $a(u, u) = 0$? $(u(x) \in V, \quad u(0) = 0)$

$$u(x) = \int_0^x u'(t)\, dt$$

$$|u(x)| = \left| \int_0^x u'(t)\, dt \right| \leq \sqrt(x) \sqrt{\int_0^x (u'(t))^2 dt} \leq \sqrt{a(u, u)}$$

Hence $a(u, u) = 0$ implies that $u(x) = 0$ pointwise.

Theorem 1.9. *Let* $V_N = span(\phi_1, \phi_2 ... \phi_N) \subset V$. *Let* u *be a weak solution of the BVP problem, i.e*

$$(f, v) = a(u, v)$$

for any $v \in V$. *If* f_N *is the projection of* f *to* V_N *and the* $u_N \in V_N$ *satisfies the equality*

$$(f_N, v_N) = a(u_N, v_N)$$

for any $v_N \in V_N$ *then* u_N *is the projection of the weak solution* u *to* V_N. *(*u_N *is the closest element from* V_N *to the weak solution* u *)*

Proof. Given that

$$(f, v) = a(u, v)$$

holds for any $v \in V$ we obtain that

$$(f, v_N) = a(u, v_N)$$

for any $v_N \in V_N$. We also given that

$$(f_N, v_N) = a(u_N, v_N).$$

Hence

$$0 = (f - f_N, V_N) = a(u - u_N, v_N)$$

and u_N is the projection of the weak solution u to the subspace V_N (We used the bilinear form $a(u, v)$ as a new inner product in V instead of the inner product $(u, v) = \int_0^1 uvdx + \int_0^1 u'v'dx$ inherited from the Sobolev space but one can show that V is complete with respect to this new inner product.) □