Mat1062: Introductory Numerical Methods for PDE

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1 Ownership

These notes are the joint property of Rob Almgren and Mary Pugh.

2 Finite Element Methods

For finite element methods we need a Hilbert Space. This is a complete[1] vector space which has an inner product and a countable basis.

To start, we will focus on the two-point boundary value problem

$$\begin{cases} -u''(x) = f(x) \quad \forall\, x \in (0,1) \\ u(0) = 0 \\ u'(1) = 0 \end{cases} \tag{1}$$

Given a continuous f, the solution is easy to find:

$$u(x) = F(x) + mx + b$$

where $F''(x) = f(x)$ and the integration constants m and b are chosen to satisfy the boundary conditions. We say that u is a *classical solution* if u has two derivatives, if u'' is continuous on $(0,1)$ and if the boundary value

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[1] A space is complete if any convergent sequence converges to something *in* the space. For example, the real numbers are complete because any convergent sequence of real numbers converges to a real number. (They can't converge to a complex number, for example.) The rational numbers are not complete because there are convergent sequences of rational numbers that don't converge to a rational number. (One can construct a sequence of rational numbers that converges to $\sqrt{2}$, for example.)

problem is satisfied. We would like to define a weaker type of solution, one which is based on a variational formulation. (Because the problem is exactly solvable all of this may feel a little pointless. But the point is: we will be studying a simple case using methods that generalise to harder problems.)

To understand how we might formulate a weak solution, assume that u is a classical solution and v is a continuous function whose first derivative v' is continuous on $(0,1)$. Then

$$\begin{aligned} \int_0^1 f(x)v(x)\,dx &= -\int_0^1 u''(x)v(x)\,dx \\ &= \int_0^1 u'(x)v'(x)\,dx - u'(1)v(1) + u'(0)v(0) \\ &= \int_0^1 u'(x)v'(x)\,dx + u'(0)v(0) \end{aligned}$$

Above, we used that $u'(1) = 0$. This calculation helps us in choosing what Hilbert Space we will look in for our solution. Specifically, we choose the Hilbert space

$$\begin{aligned} V = \{ & v(x) \text{ real-valued functions on } \mathbb{R} \\ & \left| \int_0^1 v(x)^2\,dx < \infty, \int_0^1 v'(x)^2\,dx < \infty, v(0) = 0 \right\} \end{aligned} \qquad (2)$$

We then define a *weak solution* as: if $f \in L^2([0,1])$ then u is a weak solution of the boundary value problem (1) if $u \in V$ and $\int u'v' = \int fv$ for all $v \in V$.

Before proceeding, we ask if the Hilbert space V makes sense. It's a subspace of $H^1([0,1])$ which is a Hilbert space. And so it will be a Hilbert space in its own right. The only thing to worry about is whether it makes sense to specify the value of v at a point. If all we knew about a function, v, was that $\int v^2 < \infty$ then it would not make sense to ask what v equals at a particular point. However, it turns out that if $\int v'^2 < \infty$ then this forces v to be continuous on $[0,1]$. And so it does make sense to talk about the value of v at a point. Note that the space V has one of the two boundary conditions built in to it. And so if $u \in V$ it automatically satisfies $u(0) = 0$. A natural question is whether or not we know that $u'(1) = 0$. (Note: if all we know is that $\int u'^2 < \infty$ then we can't know pointwise information like $u'(1) = 0$. But if we happen to know that u' is continuous then we would hope that $u'(1) = 0$ follows somehow.)

Now that we're satisfied with the definition of V we ask whether

$$\int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx, \qquad \forall\, v \in V$$

makes sense. Specifically, we need that each of these integrals is finite. This follows via the Schwartz inequality:

$$\left|\int_0^1 f(x)v(x)\,dx\right| \le \sqrt{\int_0^1 f(x)^2\,dx}\sqrt{\int_0^1 v(x)^2\,dx} < \infty,$$

where we used that $f \in L^2([0,1])$ and that $v \in V$. Similarly,

$$\left|\int_0^1 u'(x)v'(x)\,dx\right| \le \sqrt{\int_0^1 u'(x)^2\,dx}\sqrt{\int_0^1 v'(x)^2\,dx} < \infty$$

where we used that $u, v \in V$.

Now that we're satisfied with our definition of weak solution, we ask the natural question: "If u is smooth and is a weak solution does this imply u is a classical solution?" The answer is "yes". (If the answer were "no" then we'd really have to question our definition of weak solution.) Before proving this, we introduce some notation:

$$a(u,v) := \int_0^1 u'(x)v'(x)\,dx \qquad \langle f, v\rangle := \int_0^1 f(x)v(x)\,dx$$

Theorem *Assume* f *and* u'' *are continuous on* $[0,1]$*. Let the space* V *be as defined in (2). If* $u \in V$ *and* $a(u,v) = \langle f, v\rangle$ *for all* $v \in V$ *then* u *is a classical solution of the boundary value problem (1).*

Proof Choose $v \in V$ such that v' is continuous on $[0,1]$. Then

$$\begin{aligned}
&\int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx \\
&\quad\Longrightarrow\quad -\int_0^1 u''(x)v(x)\,dx + u'(1)v(1) - u'(0)v(0) = \int_0^1 f(x)v(x)\,dx \\
&\quad\Longrightarrow\quad -\int_0^1 u''(x)v(x)\,dx + u'(1)v(1) = \int_0^1 f(x)v(x)\,dx \\
&\quad\Longrightarrow\quad u'(1)v(1) = \int_0^1 \left(f(x) + u''(x)\right) v(x)\,dx \qquad (3)
\end{aligned}$$

The identity (3) holds for any $v \in V$ that has a continuous first derivative. We now show that this implies that the continuous function $f + u''$ is identically zero on $[0, 1]$. Assume that $f + u''$ isn't identically zero on $[0, 1]$. Because $f + u''$ is continuous on $[0, 1]$ this would imply we could find an interval $(x_0, x_1) \subset [0, 1]$ on which $f + u''$ is positive. (If we can find no such interval then we can find an interval on which $f + u''$ is strictly negative.) We use this interval to construct a specific test function

$$v(x) = \begin{cases} (x - x_0)^2 (x - x_1)^2 & x_0 \leq x \leq x_1 \\ 0 & \text{otherwise} \end{cases}$$

For this test function, (3) becomes

$$0 = \int_0^1 (f(x) + u''(x))\, v(x)\, dx = \int_{x_0}^{x_1} (f(x) + u''(x))\, v(x)\, dx > 0$$

which is impossible. (Note: if $f + u''$ were strictly negative on the interval then the above would have yeilded $0 < 0$ which is again impossible.) This proves that $-u''(x) = f(x)$ at all points in $[0, 1]$. As a result, identity (3) reduces to

$$u'(1)v(1) = 0$$

for any $v \in V$ that has a continuous first derivative. For example, it holds for $v(x) = x$ resulting in $u'(1)v(1) = u'(1) = 0$. There was nothing really special about this choice — any $v \in V$ which has continuous first derivative (so that (3) applies) and has $v(1) \neq 0$ would have resulted in $u'(1) = 0$. This finishes the proof.

Essential boundary conditions and natural boundary conditions We have boundary conditions at $x = 0$ and $x = 1$. The boundary condition at $x = 0$ is $u(0) = 0$. We built this boundary condition into the function space V. We're looking for a solution in the function space V — if we find a solution in V it will automatically satisfy the left boundary condition because that's part of the space's defintiion. When a boundary condition is part of how the function space is defined, the boundary condition is called *essential*. We just found that if a solution of the variational formulation is smooth enough (i.e. u'' is continuous) then it will satisfy $u'(1) = 0$. When a boundary condition follows from the variational formulation in this way, the boundary condition is called *natural*.

We now have a weak formulation that we are happy with:

$$u \in V, \qquad a(u,v) = \langle f, v\rangle \quad \forall\, v \in V \tag{4}$$

Another way to understand this would be

$$u \in V, \qquad \int_0^1 u'(x)v'(x) - f(x)v(x)\; dx = 0 \quad \forall\, v \in V$$

which is the same thing as looking at the first variation in V of the action functional

$$\mathcal{S}(u) = \int_0^1 \tfrac{1}{2}u_x^2(x) - f(x)u(x)\; dx$$

with Lagrangian $1/2\ u_x^2 - fu$. And so if you remember Hamilton's principle from classical mechanics you would see how the action functional would lead to the first variation would lead to the Euler-Lagrange equation which is $-u_{xx} = f$. If you're trying to solve a problem which has a Lagrangian then it is natural to use a finite element approach to implement Hamilton's principle using whatever family of basis functions you've chosen.

Finally, we note that $a(u,v)$ is actually an inner product on V. To be an inner product we need to check the following

1. $a(u,v) = \overline{a(v,u)} \quad \forall u, v \in V$. This holds automatically because u and v are real-valued functions.
2. $a(u, v+w) = a(u,v) + a(v+w) \quad \forall u, v, w \in V$. This holds because $(v+w)' = v' + w'$.
3. $a(\lambda u, v) = \lambda a(u,v) \quad \forall u, v \in V, \forall \lambda \in \mathbb{R}$. This holds because you can pull constants out of integrals.
4. $a(u,u) \geq 0 \quad \forall u \in V$ This holds because $u'^2(x) \geq 0$ for all x.
5. $a(u,u) = 0 \iff u = 0$. It's clear that $u = 0$ implies $a(u,u) = 0$. We can rigorously prove the other direction as well. Here is a nearly-rigorous proof of why $a(u,u) = 0$ implies $u = 0$. Fix $x \in [0,1]$. Then

$$u(x) = u(0) + \int_0^x u'(y)\; dy \qquad \Longrightarrow \qquad u(x) = \int_0^x u'(y)\; dy.$$

Applying the Schwartz inequality, for $x \in (0,1]$

$$|u(x)| \leq \left| \int_0^x u'(y)\ dy \right| \leq \sqrt{x} \sqrt{\int_0^x (u'(y))^2\ dy} \leq \sqrt{a(u,u)}.$$

And so $a(u,u) = 0$ implies $u = 0$ pointwise.