Mat1062: Introductory Numerical Methods for PDE

Mary Pugh

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1 Ownership

These notes are the joint property of Rob Almgren and Mary Pugh.

2 Convergence, Consistency, and Stability examples

We wish to time-step the ODE:

$$\frac{dU}{dt} = f\big(U(t), t\big) \quad \text{for } t \in [0, T], \qquad \text{with} \quad U(0) = U_0.$$

Choose an integer M and a final time T, thus determining the time step $k = T/M > 0$. Define time "levels" $t_n = nk$, for $n = 0, 1, \ldots, M$. We want to construct a sequence of points in $\mathbb{R}^N$: $u^0, u^1, \ldots, u^M$, so that $u^n \approx U(nk)$ when k is small. Here u^n is the solution of the fully discrete model, and $U(nk)$ is the exact solution of the ODE, evaluated at the discrete time points.

Consider a linear partial differential equation of the form

$$P(\partial_t, \partial_x)u(x,t) = f(x,t) \tag{1}$$

which is of first order in the time derivative. Consider a fixed one-step finite difference scheme for the equation

$$P_{k,h}u = f. \tag{2}$$

Given initial data $\{u_j^0\}$ where $j \in \mathbb{Z}$ we denote the solution of the scheme (2) by u_j^n where j ranges over $\mathbb{Z}$ and n ranges over $\mathbb{N}$. Recall the definition of convergence:

Let $u(x,t)$ *be the solution of the PDE (1) with initial data* $u_0(x)$. *Given a spatial grid spacing* h, *generate approximate initial data* $\{u_j^0\}$ *in such a way that as* jh *converges to* x *the approximate initial data* u_j^0 *converges to* $u_0(x)$. *For each fixed* h *and* k, *let* u_j^n *be the resulting solution of the one-step finite difference scheme (2). The scheme is* *convergent* *if* u_j^n *converges to* $u(x,t)$ *as* (jh, nk) *converges to* (x,t) *as* h *and* k *converge to 0.*

Recall the definition of consistency:

Given a partial differential equation, $Pu = f$, *and a finite difference scheme,* $P_{k,h}u = f$, *we say the finite difference scheme is* *consistent* *with the partial differential equation if for any smooth* $\phi(x,t)$

$$P\phi - P_{k,h}\phi \to 0, \qquad \text{as } k, h \to 0,$$

the convergence being pointwise convergence at each grid point.

As an example, consider the advection equation $u_t + u_x = 0$ for which

$$P = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$$

and the difference operator $P_{k,h}$ given by the forward-time forward-space scheme

$$P_{k,h}\phi = \frac{\phi_j^{n+1} - \phi_j^n}{k} + \frac{\phi_{j+1}^n - \phi_j^n}{h}$$

Here, ϕ is a smooth function and $\phi_j^n = \phi(jh, nk)$. We approximate all the terms in $P_{k,h}\phi$ with Taylor series in x and t about the point $(x_j, t_n) = (jh, nk)$.

$$\begin{aligned}
\phi_j^{n+1} &= \phi_j^n + k\phi_t(x_j,t_n) + \tfrac{1}{2}k^2\phi_{tt}(x_j,t_n) + O(k^3) \\
\phi_{j+1}^n &= \phi_j^n + h\phi_x(x_j,t_n) + \tfrac{1}{2}h^2\phi_{xx}(x_j,t_n) + O(h^3)
\end{aligned}$$

Plugging this in,

$$P_{k,h}\phi(x_j,t_n) = \phi_t + \phi_x + \tfrac{1}{2}k^2\phi_{tt} + \tfrac{1}{2}h\phi_{xx} + O(k^2) + O(h^2)$$

where the derivatives are all evaluated at (x_j, t_n). Thus

$$P\phi - P_{k,h}\phi = -\tfrac{1}{2}k^2\phi_{tt} - \tfrac{1}{2}h\phi_{xx} + O(k^2) + O(h^2) \quad \to \quad 0$$

as $(h, k) \to (0, 0)$. Therefore this scheme is consistent.

Above, you have to think things through a little carefully. Naively, if $(x_j, t_n) = (jh, nk)$ where j and n are fixed then $h, k \to 0$ would force $x_j, t_n \to 0$. This isn't what's intended above. What you should understand is: Let $\phi(x, t)$ be a smooth function on $\mathbb{R} \times [0, \infty)$. Pick (x_0, t_0) in that region. Without loss of generality, assume $x_0 > 0$. For each n and j in $\mathbb{N}$, take $h_j := x_0/j$ and $k_n := t_0/n$. By construction, $(x_0, t_0) = (jh_j, nt_n)$ and

$$\begin{aligned} P\phi(x_0, t_0) - P_{k_n,h_j}\phi(x_0, t_0) &= -\tfrac{1}{2}k_n^2\phi_{tt}(x_0, t_0) - \tfrac{1}{2}h_j\phi_{xx}(x_0, t_0) \\ &\quad + O(1/n^2) + O(1/j^2) \quad \to \quad 0 \end{aligned}$$

as $j, n \to \infty$. Note that j and n have to go to infinity simultaneously. Also note that the definition of consistency had nothing to do with ϕ being a solution of the PDE. It had to do with how the approximation of the operator acts on smooth functions. Note that if $Pu = 0$ has a smooth solution and we take $\phi(x, t) = u(x, t)$ where u a solution then controlling $Pu - P_{k,h}u$ is the same thing as controlling $P_{k,h}u$. Which is what we did when finding the local truncation error. (Translation: when checking that a scheme is consistent, you get the local truncation error for free.)

Consistency implies that the solution of the partial differential equation, if it is smooth, is an approximate solution of the finite difference scheme. (That is, if $Pu = 0$ then $P_{k,h}u$ goes to zero as $h, k \to 0$ and so u is an approximate solution of the discrete scheme.) Convergence means that a solution of the finite difference scheme approximates a solution of the partial differential equation. It is natural to ask whether consistency is sufficient for a scheme to be convergent. Consistency is certainly necessary for convergence, but is not sufficient.

The following example gives a scheme which is consistent but not convergent. Take the above forward-time forward-space scheme for $u_t + u_x = 0$. You can rewrite it as

$$u_j^{n+1} = u_j^n - \frac{k}{h}\left(u_{j+1}^n - u_j^n\right) = (1 + \lambda)u_j^n - \lambda u_{j+1}^n$$

where $\lambda = k/h$. Now, take initial data

$$u_0(x) = \begin{cases} x^4 & \text{if } x < 0 \\ 0 & \text{otherwise} \end{cases}$$

The solution is $u(x,t) = u_0(x-t)$ and so for any $t > 0$ we have $u(x,t) = (x-t)^4$ if $x < t$ and 0 otherwise. And so, if $t > 0$ there are positive x at which $u(x,t) > 0$. For the difference scheme, take initial data

$$u_j^0 = \begin{cases} (jh)^4 & \text{if } j < 0 \\ 0 & \text{otherwise} \end{cases}$$

Because $u_j^{n+1} = (1+\lambda)u_j^n - \lambda u_{j+1}^n$ it is clear that if $j \geq 0$ then $u_j^0 = 0$ and $u_j^n = 0$ for all $n > 0$. This is because u_j^{n+1} is determined by data below and to the right.

Clearly, if $x_j > 0$ it is impossible for u_j^n to converge to $u(x_j, t_n) > 0$ as $h, k \to 0$ and so the scheme is not convergent. Note that it's not a problem with the solution $u(x,t)$; this solution is smooth enough for the expansions we did in testing the scheme's consistency to be valid.

Recall what stability means for a *homogeneous* finite difference scheme

A finite difference scheme $P_{k,h}u = 0$ for a first-order equation is stable if there is an integer J and positive numbers h_0 and k_0 such that for any positive time T there is a constant C_T such that

$$h \sum_{j=-\infty}^{\infty} |u_j^n|^2 \leq C_T \sum_{l=0}^{J} h \sum_{j=-\infty}^{\infty} |u_j^l|^2$$

for $0 \leq nk \leq T$, $0 < h \leq h_0$, and $0 < k \leq k_0$.

Note that if we introduce the discrete analogue of the $L^2(\mathbb{R})$ norm

$$||w||_h = \left(h \sum_{j=-\infty}^{\infty} |w_j|^2 \right)^{1/2}$$

then the stability constraint can be written as

$$||u^n||_h^2 \leq C_T \sum_{l=0}^{J} ||u^l||_h^2$$

and hence

$$||u^n||_h \leq \widetilde{C}_T \sum_{l=0}^{J} ||u^l||_h \tag{3}$$

for some $\widetilde{C}_T$.

We will usually use Fourier methods to check the stability of a scheme rather than trying to show directly that inequalty (3) holds. But here is a simple example where we can show it directly. Consider the scheme

$$u_j^{n+1} = \alpha_{h,k} u_j^n + \beta_{h,k} u_{j+1}^n$$

We show that if there is some h_0 and k_0 so that $h \le h_0$ and $k \le k_0$ implies $|\alpha_{h,k}| + |\beta_{h,k}| \le 1$ then this scheme is stable. In the following, I will suppress the h and k dependence of the coefficients $\alpha_{h,k}$ and $\beta_{h,k}$.

$$\begin{aligned} h \sum_{j=-\infty}^{\infty} |u_j^{n+1}|^2 &= h \sum_{j=-\infty}^{\infty} |\alpha u_j^n + \beta u_{j+1}^n|^2 \\ &= h \sum_{j=-\infty}^{\infty} |\alpha|^2 |u_j^n|^2 + |\alpha||\beta| 2 |u_j^n||u_{j+1}^n| + |\beta|^2 |u_{j+1}^n|^2 \\ &\le h \sum_{j=-\infty}^{\infty} |\alpha|^2 |u_j^n|^2 + |\alpha||\beta| (|u_j^n| + |u_{j+1}^n|)^2 + |\beta|^2 |u_{j+1}^n|^2 \\ &= h \sum_{j=-\infty}^{\infty} (|\alpha| + |\beta|)^2 |u_j^n|^2 \le h \sum_{j=-\infty}^{\infty} |u_j^n|^2 \end{aligned}$$

(In the last step, I used my assumption that $h \le h_0$ and $k \le k_0$ implies $|\alpha_{h,k}| + |\beta_{h,k}| \le 1$.) One can repeat this argument all the way down to 0, resulting in

$$h \sum_{j=-\infty}^{\infty} |u_j^{n+1}|^2 \le h \sum_{j=-\infty}^{\infty} |u_j^0|^2$$

and so $C_T = 1$ and $J = 0$ works for the stability inequality (3).

The stability inequality (3) states that the L^2 norm of the discrete solution at level n is bounded by some constant times the sum of the L^2 norms of the discrete solution at the first $J + 1$ levels. For one-step schemes, we will see that one can take $J = 0$ — all that matters is the L^2 norm of the discrete initial data.

3 the Lax-Richtmyer Equivalence Theorem

From your PDE course, you may recall that there are various definitions of what it means for an initial value problem to be well-posed. If one considers linear, homogenous problems then L^2 well-posedness would be defined as:

The initial value problem for the first-order partial differential equation $Pu = 0$ *is well-posed if for any time* $T > 0$ *there is a constant* C_T *such that any solution* $u(x,t)$ *satisfies*

$$\sqrt{\int_{-\infty}^{\infty} |u(x,t)|^2 \, dx} \leq C_T \sqrt{\int_{-\infty}^{\infty} |u(x,0)|^2 \, dx} \tag{4}$$

for $0 \leq t \leq T$.

Indeed, if you can show that the inequality (4) holds then uniqueness of solutions and continuous dependence of solutions on initial data follows immediately. It turns out that if the one-step finite difference scheme arises from a well-posed PDE initial value problem then consistency, convergence, and stability are all closely related:

Lax-Richtmyer Equivalence Theorem A consistent one-step scheme for a well-posed initial value problem for a partial differential equation (1) is convergent if and only if it is stable.