

# Mat1062: Introductory Numerical Methods for PDE

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## 1 Ownership

These notes are the joint property of Rob Almgren and Mary Pugh.

## 2 Hyperbolic equations

Now we will talk about *hyperbolic* equations. As you may recall, in the first lecture we introduced the basic “trichotomy” around which we structured the course. We take the particular case

$$a u_{xx} + 2b u_{xy} + c u_{yy} = \text{lower order terms.} \quad (1)$$

We have taken two dimensions for simplicity; higher dimensions are straightforward. This classification really only applies to *quasi-linear, second-order, scalar* PDEs, but we apply the ideas much more generally.

The coefficients  $a, b, c$  define a matrix  $Q = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$  and an associated quadratic form  $G(p) = p^T Q p$  for  $p \in \mathbb{R}^2$ .

- $Q$  is positive definite if  $G(p) > 0$  for all  $p \in \mathbb{R}^2$ . Similarly, it's negative definite if  $G(p) < 0$ . If  $Q$  is positive or negative definite then the equation (1) is elliptic. Example  $u_{xx} + u_{yy} = 0$ , with  $Q = I$ . Solutions are determined completely by the boundary data; in effect the propagation speed is infinite since all points are coupled to all other points.

- If  $G(p) \geq 0$  for all  $p \in \mathbb{R}^2$  and there is some  $p_0$  such that  $G(p_0) = 0$  then equation (1) is parabolic. (Similarly if  $G(p) \leq 0$  for all  $p$ ...) In this case we need lower-order terms in equation (1) to get a reasonable problem. Example  $u_t = u_{xx}$  (changing  $y$  to  $t$ ), with  $Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ . As discussed, the speed of propagation is a complicated concept; in a strict sense it is infinite but in a more realistic sense information propagates with “diffusive scaling”  $x \sim t^{1/2}$ . In homework we explored a nonlinear version that gave anomalous scaling  $x \sim t^{1/3}$ .
- A hyperbolic equation has  $G(p) > 0$  for some  $p$  and  $G(p) < 0$  for others. If  $G(p_0) = 0$  then  $p_0$  determines the direction and speed of propagation. For example if  $u_{yy} - c^2 u_{xx} = 0$  then

$$Q = \begin{pmatrix} -c^2 & 0 \\ 0 & 1 \end{pmatrix} \implies G(p) = 0 \text{ for } p = \begin{pmatrix} 1 \\ c \end{pmatrix}, \begin{pmatrix} 1 \\ -c \end{pmatrix}.$$

Using the quadratic form  $G(p)$ , it's possible to introduce new coordinates

$$\tilde{x}(x, y), \quad \tilde{y}(x, y)$$

so that the PDE (1) is transformed into one of

$$u_{\tilde{x}\tilde{x}} + u_{\tilde{y}\tilde{y}} = \text{lower order terms}$$

$$u_{\tilde{x}} = u_{\tilde{y}\tilde{y}} + \text{lower order terms}$$

$$u_{\tilde{x}\tilde{x}} = u_{\tilde{y}\tilde{y}} + \text{lower order terms}$$

That is, if the PDE (1) is elliptic then it's Poisson's equation after a change of coordinates, if it's parabolic then it's the diffusion equation after a change of coordinates, and if it's hyperbolic then it's the wave equation after a change of coordinates. This is the reason why introductory PDE courses spend so much time on the diffusion equation, the wave equation, and Poisson's equation — they're fairly universal. That said, introductory PDE courses usually have very simple “lower order terms” on the right-hand side of the equation (like zero or a simple function  $f(x, t)$ ). Other types of right-hand sides can already make life quite interesting.

It is frequently possible to write a second-order differential equation as a system of first-order equations. For example, this is always possible for

an ordinary differential equation. It is not always possible for a PDE but usually is, since most physical problems are really based on first derivatives. For a second-order PDE, we look for a first-order system of the form

$$u_t + A u_x = 0, \quad (2)$$

where  $u$  is now an  $n$ -component function of  $(x, t)$ , and  $A$  is an  $n \times n$  matrix. A system derived from a scalar second-order equation will have  $n = 2$ . I will use  $t$  and  $y$  interchangeably depending on what kind of equation we are talking about. Then the properties of the system are determined by the eigenvalues of the matrix  $A$ .

**Elliptic example** Consider the Laplace equation  $u_{xx} + u_{yy} = 0$ . Writing  $v = u_y$  and  $w = u_x$ , we get the system

$$\begin{pmatrix} v \\ w \end{pmatrix}_y + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x = 0,$$

The matrix  $A$  is *skew-symmetric* ( $A^T = -A$ ) and thus has purely imaginary eigenvalues  $\pm i$ . (This system is in fact the Cauchy-Riemann equations of complex analysis, for  $f(z) = v + iw$ .) The complex eigenvalues are typical of elliptic problems, and indicate that this reduction is not very useful. Just as we discussed in stability analysis, complex speeds of propagation correspond to exponential growth and decay of Fourier modes, and indicate that it is not a good idea to specify only "initial" data.

**Hyperbolic example** Let's do the same for the wave equation

$$u_{tt} = c^2 u_{xx}.$$

Setting  $v = u_t$  and  $w = -cu_x$ , we get the first-order system

$$\begin{pmatrix} v \\ w \end{pmatrix}_t + \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x = 0.$$

Here  $A$  is symmetric; it has real eigenvalues  $\pm c$  and eigenvectors  $(1, \mp 1)^T$ . These eigenvalues are exactly the finite speeds of propagation.

Motivated by this example, we widen our definition of hyperbolic problems. Instead of considering second-order scalar equations, we start with

systems of the form (2), of any size  $n \geq 1$ . A *linear hyperbolic system* is one whose matrix  $A$  has a full set of eigenvectors and only real eigenvalues. This is often attained by  $A$  being *symmetric* but not always.

Soon we will generalize these problems to replace  $Au$  by  $F(u)$  where  $F$  is a nonlinear function  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ . We denote  $A(u) = \nabla_u F$ , and define a *nonlinear hyperbolic conservation law* to be a system of the form  $u_t + F(u)_x = 0$ , whose gradient matrix  $A$  has real eigenvalues for all relevant values of  $u$ . (Various kinds of degeneracy are also possible and lead to interesting problems.) Linear and nonlinear hyperbolic systems are easily defined in more than one space dimension ( $u(x, y, t)$ , etc).

The theory of nonlinear hyperbolic systems is one of the centerpieces of applied mathematics. These systems are of great importance in applications, but also it is one of the few types of nonlinear problems that we really understand. In particular, the concept of *weak solution* was first fully developed in this context. Conservation is the key idea that makes it work.

## Linear scalar problems

The simplest hyperbolic equation is the linear scalar problem

$$u_t + a u_x = 0, \quad (3)$$

with initial data  $u(x, 0) = u_0(x)$ , and  $a$  constant. The solution is clearly

$$u(x, t) = u_0(x - at) \quad (4)$$

as you may check by differentiation. That is, to know what the solution value  $u$  is at a point  $(x, t)$ , look back to  $t = 0$ , but at a location  $x = at$ . Information travels along the straight *characteristic lines*  $x = at + \text{const}$ .

It may be that  $u_0(x)$  is not a differentiable function—has a discontinuity, say—so that the function  $u(x, t)$  defined by (4) does not have derivatives  $u_t$  or  $u_x$ , and hence it is not clear what the PDE (3) means. But in the same way that we defined solutions of a second-order elliptic problem as the minimizer of an energy involving only first derivatives, we decide that (4) will be the *weak solution* of (3) whether or not  $u_0$  has the required derivatives. This is based on our interpretation of the PDE (3) as representing a physical effect (convection) rather than just a relationship among derivatives.

**Boundary conditions** Take the domain to be  $x > 0$ , with Dirichlet boundary data  $u(0, t) = g(t)$  on  $x = 0$ . Does there exist a solution having initial data  $u_0(x)$  and this boundary data? Answer: yes, if and only if  $a > 0$ ; this may be understood by tracing the characteristics in the  $(x, t)$ -plane.

If  $a > 0$ , then characteristics leaving  $x > 0$ ,  $t = 0$  head off to positive  $x$ . Further, the characteristics leaving the boundary  $x = 0$ ,  $t > 0$  also head into the positive half-line, and they fill in the information missing from the initial data. For any point  $(x, t)$  with  $x > 0$ ,  $t > 0$ , we can follow back the trajectory  $x - at = \text{const}$  to find a unique data point that determines the solution along that characteristic.

But if  $a < 0$ , then characteristics leaving the initial line  $t = 0$  hit the boundary  $x = 0$ ,  $t > 0$ . Unless  $u_0(-at) = g(t)$  for all  $t > 0$ , then the values at the two ends of the characteristics are *inconsistent*. There is *no* function  $u(x, t)$  that can take both endpoint values that are imposed at the two ends of the characteristic segment.

**Nonconstant coefficients** Suppose that  $a = a(x, t)$  is a function of  $(x, t)$ . The equation may be written in one of two ways: either

$$u_t + a(x, t) u_x = 0 \quad (5)$$

or

$$u_t + (a(x, t) u)_x = 0. \quad (6)$$

The second form is conservative, so that  $\int_a^b u(x, t) dx$  changes only due to fluxes across the endpoints. If everything is smooth, we may expand (6) as

$$u_t + a(x, t) u_x = -a_x u,$$

so (6) is like (5), with an extra source term that changes  $u$ .

We may write (5) as

$$\frac{d}{dt} u(\xi(t), t) = 0$$

as long as

$$\frac{d\xi}{dt} = a(\xi(t), t). \quad (7)$$

Thus  $u(x, t)$  is constant along *curved* characteristic paths, defined by the ODE (7). The value of the solution  $u(x, t)$  for  $t > 0$  is found by following back the curve  $\xi(t)$  to its initial value  $u_0(\xi(0))$ . Note that (unless  $a(\xi, t)$

is a really bad function) this evolution is uniquely defined forward and backward in time. Thus characteristics never cross, and this procedure uniquely defines the solution  $u(x, t)$  everywhere. Boundary conditions may be handled as above.

The conservative form (6) may be written

$$\frac{d}{dt}u(\xi(t), t) = -a_x(\xi(t), t) u(\xi(t), t).$$

This equation also propagates data along the same characteristic curves, but with a nonzero right-hand side that modifies the value of  $u$  as the characteristics come closer together or move apart, so as to preserve conservation.

### Systems of equations

Now consider  $u_t + Au_x = 0$  where  $A$  is a given  $n \times n$  diagonalizable matrix, constant for now. Let  $P = (v_1, \dots, v_n)$  be the eigenvectors of  $A$ , and  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$  its eigenvalues, so that  $AP = P\Lambda$ . Then writing  $u(x, t) = Pw(x, t)$ , we see that

$$w_t + P^{-1}AP w_x = 0.$$

Since  $P^{-1}AP = \Lambda$ , this is equivalent to the  $n$  scalar equations

$$(w_j)_t + \lambda_j (w_j)_x = 0, \quad j = 1, \dots, n.$$

That is, the  $n \times n$  linear system (2) has  $n$  characteristic speeds, given by the eigenvalues of  $A$ . The eigencomponents  $w_j$  propagate without interaction (unless there are boundaries in which case the boundary conditions could do things like transfer energy from one eigenvector to others).

If  $A = A(x, t)$ , then as above, the characteristic speeds  $\lambda(x, t)$  and eigenvectors depend on space and time. We then get the system

$$w_t + \Lambda w_x = B(x, t) w, \quad B = -P^{-1}(P_t + AP_x).$$

The linear term on the right side couples the different eigenmodes together. For example, this is how linear waves reflect from gradients in material density or wave speed.