Mat1062: Introductory Numerical Methods for PDE

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1 Ownership

These notes are the joint property of Rob Almgren and Mary Pugh.

2 Weak solutions

Assuming we have a function $u(x,t)$ that satisfies

$$u_t + f(u)_x = 0. \tag{1}$$

we can integrate over a fixed region in space-time interval $[\alpha,\beta] \times [t_0, t_1]$ to write

$$\int_{t_0}^{t_1} \int_{\alpha}^{\beta} u_t(x,t)\, dx\, dt + \int_{t_0}^{t_1} \int_{\alpha}^{\beta} f(u(x,t))_x\, dx\, dt = 0.$$

It then follows that

$$\int_{\alpha}^{\beta} u(x,t_1)\, dx = \int_{\alpha}^{\beta} u(x,t_0)\, dx + \int_{t_0}^{t_1} f(u(\alpha,t))\, dt - \int_{t_0}^{t_1} f(u(\beta,t))\, dt. \tag{2}$$

That is, the amount of "stuff" in $[\alpha,\beta]$ at time t_1 is equal to the amount of stuff in $[\alpha,\beta]$ at time t_0 plus the amount of stuff that flowed into the interval through the $x = \alpha$ boundary between times t_0 and t_1 minus the amount of stuff that flowed out through the $x = \beta$ boundary between times t_0 and t_1.

A smooth solution of $u_t + (f(u))_x = 0$ will satisfy equation (2) for all $\alpha < \beta$ and all $t_0 < t_1$. If a function $u(x,t)$ satisfies equation (2) for all $\alpha < \beta$ and all $t_0 < t_1$ then we call u a *weak solution*. Most physical systems are originally derived in the integral form (2); the differential form (1) is derived under an assumption of smoothness. It is therefore reasonable to take (2) as the rule that must be preserved for discontinuous solutions when (1) does not apply. The differential form

$$g(u)_t \; + \; f(u)_x \; = \; 0, \tag{3}$$

has an analogous integral form with weak solutions; rather than considering $\int_\alpha^\beta u$ one considers $\int_\alpha^\beta g(u)dx$ and so on.

An alternate, but equivalent, formulation of weak solution with initial data $u_0(x)$ at time $t = 0$ is

$$\int_{\mathbb{R}} \int_0^\infty u(x,t)\; \phi_t(x,t) + f(u(x,t))\; \phi_x(x,t)\; dx\; dt = -\int_{\mathbb{R}} u_0(x)\; \phi(x,0)\; dx \tag{4}$$

where $\phi(x,t)$ is a differentiable function with compact support in $\mathbb{R} \times [0, \infty)$. If (4) holds for all such test functions ϕ then we say u is a weak solution.

Shock waves are weak solutions. The need to understand shock waves is one of the main reasons to consider weak solutions in the first place. The shock speed is dermined from the integral formulation (2) as follows. Suppose that $u(x,t)$ is smooth except for a simple jump discontinuity across the line $x = \xi(t)$ in the (x,t)-plane. Let

$$u_L(t) \; = \lim_{x \to \xi(t)^-} u(x,t), \qquad u_R(t) \; = \lim_{x \to \xi(t)^+} u(x,t)$$

be the values on the two sides of the discontinuity. Then by taking $t_1 \to t_0$ in the integral form (2) we find that $u(x,t)$ is a weak solution if it satisfies

$$\frac{d}{dt} \int_\alpha^\beta u(x,t)\, dx \; = \; -f(u)\big|_{x=\alpha}^{\beta} \; = \; f\big(u(\alpha,t)\big) \; - \; f\big(u(\beta,t)\big) \tag{5}$$

for all α, β, and t. Fix a time t and choose α and β on either side of $\xi(t)$.

Then

$$
\begin{aligned}
\frac{d}{dt}\int_\alpha^\beta u(x,t)\,dx \;&=\; \frac{d}{dt}\left(\int_\alpha^{\xi(t)} u\,dx \;+\; \int_{\xi(t)}^\beta u\,dx\right) && (6)\\
&= \xi'(t)\;u_L(t) + \int_\alpha^{\xi(t)} u_t(x,t)\;dx && (7)\\
&\qquad - \xi'(t)\;u_R(t) + \int_{\xi(t)}^{\alpha} u_t(x,t)\;dx && (8)\\
&= \xi'(t)\;u_L(t) - \int_\alpha^{\xi(t)} (f(u(x,t)))_x\;dx && (9)\\
&\qquad - \xi'(t)\;u_R(t) - \int_{\xi(t)}^{\alpha} (f(u(x,t)))_x\;dx && (10)\\
&=\; \xi'(t)\,u_L(t) \;-\; f(u)\Big|_\alpha^{\xi(t)} \;-\; \xi'(t)\,u_R(t) \;-\; f(u)\Big|_{\xi(t)}^{\beta}\\
&=\; \xi'(t)(u_L - u_R) \;-\; \big(f(u_L) - f(u_R)\big) \;-\; f(u)\Big|_\alpha^\beta && (11)
\end{aligned}
$$

In going from (6) to (7) and (8), I assumed that $\xi(t)$ is differentiable in time and then used the fundamental theorem of Calculus. This is valid because u is a continuous function on the interval $[\alpha, \xi(t)]$ (if extended to have the value $u_L(t)$ at $\xi(t)$) and on $[\xi(t), \beta]$ (if extended to have the value $u_R(t)$ at $\xi(t)$). In going from (7) to (9) and from (8) to (10), I used that $u(x,t)$ is, by assumption, a smooth solution of the PDE on the intervals $(\alpha, \xi(t))$ and $(\xi(t), \beta)$ and so I could use the differential form of the equation: $u_t = -(f(u))_x$. By the integral form of the equation (5),

$$f\big(u(\alpha,t)\big) \;-\; f\big(u(\beta,t)\big) = \frac{d}{dt}\int_\alpha^\beta u(x,t)\,dx$$

For this to be reconciled with (11) we see that this can happen if and only if the discontinuity satisfies the Rankine-Hugoniot condition:

$$\xi'(t) \;=\; \frac{f(u_L) - f(u_R)}{u_L - u_R}. \qquad (12)$$

That is, weak solutions can have discontinuities if the discontinuities propagate with the correct speed.

For example, a *stationary* discontinuity, with $\xi'(t) = 0$, must have $f(u_L) = f(u_R)$: since none of the conserved quantity may accumulate on a thin line, the flux in from one side must exactly balance the flux out the other side.

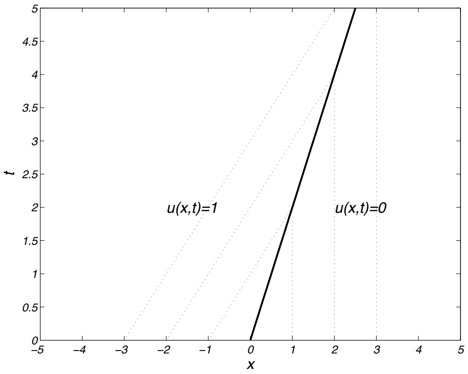

Figure 1: Bold line: the shock (discontinuity) travels on the line $t = 2x$. To the left, the solution is identically 1 and the characteristics (dotted lines) are travelling with speed 1. To the right, the solution is identically 0 and the characteristics (dotted lines) are stationary — they are vertical lines in the x-t plane. The characteristics run into the shock from both sides.

If the discontinuity is moving this corresponds to some accumulation of material: the net accumulation must balance the net influx. For example, if we have initial data

$$u_0(x) = \begin{cases} 1 & x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

and the conservation law is $u_t + (u^2/2)_x = 0$ then the shock travels with speed

$$\xi'(t) = \frac{u_R(t)^2/2 - u_L(t)^2/2}{u_R(t) - u_L(t)} = \frac{1}{2}(u_R(t) + u_L(t)) = \frac{1}{2}$$

and the weak solution is

$$u(x,t) = \begin{cases} 1 & \text{if } x < t/2 \\ 0 & \text{otherwise} \end{cases}$$

Figure 1 shows this solution in the x-t plane.

Any smooth solution of $u_t + (u^2/2)_x = 0$ is also be a solution of the conservation law $(u^2)_t + 2/3\,(u^3)_x = 0$ and vice versa. These two conservation laws have the same smooth solutions. However, they have different weak solutions! For the same step-function initial data u_0, the Rankine-Hugoniot condition for $(u^2)_t + 2/3\,(u^3)_x = 0$ is:

$$\xi'(t) = \frac{2/3\; u_R^3 - 2/3\; u_L^3}{u_R^2 - u_L^2} = \frac{2}{3}$$

and so the weak solution is

$$u(x,t) = \begin{cases} 1 & \text{if } x < 2t/3 \\ 0 & \text{otherwise} \end{cases}$$

This shows that two hyperbolic conservation laws can have the same smooth solutions but have different weak solutions.

3 A Cautionary Tale

Expand $u_t + (u^2/2)_x = 0$ and consider $u_t + uu_x = 0$ with intial data $u_0 \geq 0$. Naively, one would think that one could simply do the analogue of the explicit upwind scheme:

$$u_j^{n+1} = u_j^n - \frac{k}{h} u_j^n \left(u_j^n - u_{j-1}^n \right)$$

Take the initial data v

$$u_0(x) = \begin{cases} 1.2 & x \leq \pi/2 \\ 0.4 & \text{otherwise} \end{cases}$$

By the Rankine-Hugoniot condition

$$\xi'(t) \;=\; \frac{f(u_L) - f(u_R)}{u_L - u_R}.$$

the weak solution has a shock that travels with $\xi'(t) = .8$. However, if I use the above scheme to compute the approximate solution, I find that in the limit it produces a shock which is travelling with speed .72 (approximately); see Figure 2. As a more extreme example, consider the initial data

$$u_0(x) = \begin{cases} 1 & x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then the above explicit upwind scheme results in $u_j^n = u_j^0$ for all j and all $n > 0$ — the discrete solution is stationary while the weak solution of the conservation law has a shock that travels with speed 1/2.

This show us that we can have a scheme which is converging, but is not converging to a solution of the problem! If we'd taken smooth, positive

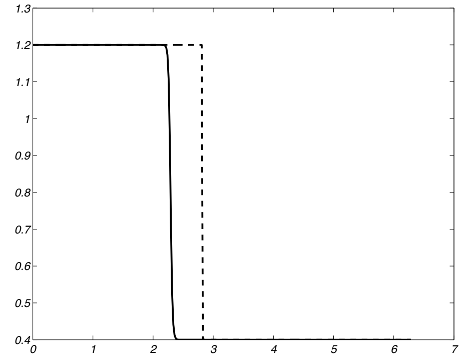

Figure 2: Bold line: the approximate solution at time $t = 1$. Dashed line: the weak solution at time $t = 1$. The space-step is $h = 2\pi/320$ and the time-step is $k = 1/320$. The approximate solution is not converging to the weak solution of the conservation law — it's moving with approximately speed 0.71. As I refine h and k I find that the limiting speed is approximately 0.72.

initial data and were computing up to a time for which the solution was still smooth, then the above scheme would be fine. It would be stable and would converge and the thing it was converging to would be a classical solution. The problem is, if we compute with non-smooth initial data or if we compute past the blow-up time (the time at which the classical solution becomes a weak solution) then although the scheme is converging to something, the thing it's converging to is not a weak solution.

What went wrong? The equation $u_t + uu_x = 0$ is not in conservation form. And so it's not too surprising that a scheme based on this form would perform differently than a scheme based on the conservative form $u_t + (u^2/2)_x = 0$. We already saw something like this when discretizing the heat equation — if the diffusivity wasn't constant then if we didn't discretize things in a "smart" way we wouldn't conserve mass.