

Mat1062: Introductory Numerical Methods for PDE

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February 10, 2009

1 Ownership

These notes are the joint property of Rob Almgren and Mary Pugh.

2 Modified equations

As usual, let us construct finite-difference methods on a grid with space step h and time step k , denoting by u_j^n our approximation to the solution value at $x = jh$, $t = nk$. We consider only the scalar first-order problem

$$u_t + au_x = 0 \tag{1}$$

with a constant.

The *modified equation* or *effective equation* is a way to understand qualitatively the effect of discretization errors on the behaviour of the solution. The idea is that we look at the leading-order truncation errors, write them in terms of derivatives of the solution, and then write the PDE that the scheme more closely approximates than the original one. This modified equation will have coefficients containing factors of h or h^2 , so as the mesh is refined they go to zero. But for any finite h , the extra terms give us insight into the behavior of the discrete solutions.

As an example, let's consider the explicit upwind scheme. We already know that if $u(x, t)$ is the solution of $u_t + au_x = 0$ and $u_j^n = u(x_j, t_n)$ then the truncation error

$$\epsilon_j^n = u_j^{n+1} - u_j^n + \mu(u_j^n - u_{j-1}^n)$$

satisfies

$$|\epsilon_j^n - k [u_t(x_j, t_n) + au_x(x_j, t_n)]| = \mathcal{O}(k^2, hk).$$

On the other hand, if $v(x, t)$ is the solution of $v_t + av_x + k/2 v_{tt} - ah/2 v_{xx} = 0$ and $v_j^n = v(x_j, t_n)$ then the truncation error

$$\epsilon_j^n = v_j^{n+1} - v_j^n + \mu(v_j^n - v_{j-1}^n)$$

satisfies

$$\left| \epsilon_j^n - k \left(v_t(x_j, t_n) + av_x(x_j, t_n) + \frac{k}{2} v_{tt}(x_j, t_n) - \frac{ah}{2} v_{xx}(x_j, t_n) \right) \right| = \mathcal{O}(k^3, h^2k).$$

This means that the discrete solution generated by explicit upwinding is actually closer to the solution of $v_t + av_x + k/2 v_{tt} - ah/2 v_{xx} = 0$ than it is to the solution of $u_t + au_x = 0$. And so to understand the properties of the discrete solution we would be better off studying properties of the solution of the v equation. In fact, we're going to find an even better equation to study. The v equation is

$$v_t = -av_x - \frac{k}{2} v_{tt} + \frac{ah}{2} v_{xx} \quad (2)$$

This means that $v_t = -av_x + \mathcal{O}(h, k)$ and hence $v_{tt} = a^2 v_{xx} + \mathcal{O}(h, k)$. As a result, if we replace the v_{tt} in equation (2) with $a^2 v_{xx}$ we will get an equation which is close to it. And so we introduce

$$w_t + aw_x = -\frac{k}{2} a^2 w_{xx} + \frac{ah}{2} w_{xx} = \frac{ah}{2} \left(1 - \frac{ak}{h} \right) w_{xx} \quad (3)$$

If $w(x, t)$ is the solution of (3) and $w_j^n = w(x_j, t_n)$ then the truncation error

$$\epsilon_j^n = w_j^{n+1} - w_j^n + \mu(w_j^n - w_{j-1}^n)$$

satisfies

$$\left| \epsilon_j^n - k \left(w_t(x_j, t_n) + aw_x(x_j, t_n) - \frac{ah}{2} \left(1 - \frac{ak}{h} \right) w_{xx}(x_j, t_n) \right) \right| = \mathcal{O}(k^3, h^2k).$$

In this way, we find that the discrete solution generated by explicit upwinding is closer to the solution of equation (3) than to the solution of $u_t + au_x = 0$: see Figure 1. We call equation (3) the *modified equation* for the explicit upwinding scheme.

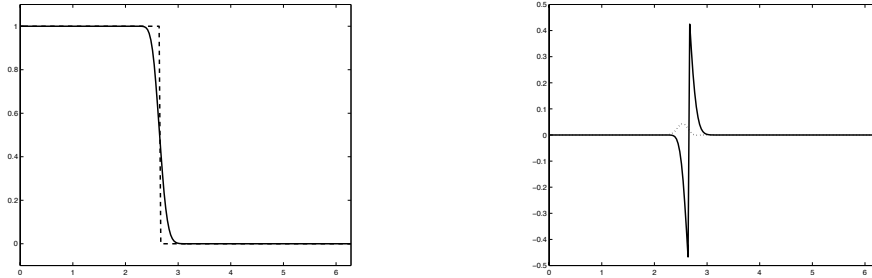


Figure 1: The parameters are as in Figures 1 and 2 of the February 5 notes. Left: The solution of the explicit upwind scheme is shown with a solid line. The exact solution of $u_t + au_x = 0$ is shown with a dashed line. The exact solution of the modified equation (3) for the explicit upwind scheme is shown with (barely visible) open circles. Right: the discrete solution minus the exact travelling wave solution is shown with a solid line. The discrete solution minus the exact solution of the modified equation is shown with a dashed line. Note that the discrete solution is much closer to the solution of the modified equation.

In a similar manner, we find that the modified equation for the Lax-Friedrichs scheme is

$$u_t + au_x = \frac{h^2}{2k} \left(1 - \left(\frac{ak}{h} \right)^2 \right) u_{xx}, \quad (4)$$

the modified equation for the Lax-Wendroff scheme is

$$u_t + au_x = \frac{h^2 a}{6} \left(\frac{a^2 k^2}{h^2} - 1 \right) u_{xxx}, \quad (5)$$

and the modified equation for the Beam-Warming scheme is

$$u_t + au_x = \frac{h^2 a}{6} \left(\frac{ak}{h} - 1 \right) \left(\frac{ak}{h} - 2 \right) u_{xxx} \quad (6)$$

2.1 First-order schemes and diffusion

Both the explicit upwind scheme and the Lax-Friedrichs scheme have modified equations that are advection diffusion equations:

$$u_t + au_x = Du_{xx}.$$

One can explicitly solve this equation by going into moving coordinates $v(y, t) = u(x+at, t)$ and then solving the resulting heat equation $v_t = Dv_{yy}$. For example, for initial data

$$u_0(x) = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{otherwise} \end{cases} \implies u(x, t) = 1 - \operatorname{erf}\left(\frac{x-at}{\sqrt{4Dt}}\right) \quad (7)$$

where

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^x e^{-t^2} dt.$$

This is a different error function than what's built in to matlab and maple. I chose this one for analytical convenience: it tends to zero as x tends to $-\infty$ and it tends to one as x tends to $+\infty$. The error function built in to maple and matlab has limit -1 at $x = -\infty$ and $+1$ at $x = +\infty$.

The diffusion constants depend on h and k :

$$D_{\text{EU}} = h \frac{a}{2} \left(1 - \frac{ak}{h}\right) = h \frac{a}{2} (1 - \mu) \quad (\text{explicit upwind}) \quad (8)$$

$$D_{\text{LF}} = \frac{h^2}{2k} \left(1 - \left(\frac{ak}{h}\right)^2\right) = h \frac{a}{2\mu} (1 - \mu^2) \quad (\text{Lax-Friedrichs}) \quad (9)$$

If one is considering the usual "refinement path" in which one chooses a fixed value of μ within the stability region and then refines h and k so that μ remains unchanged then one sees that as $h \rightarrow 0$ both diffusion coefficients D_{EU} and D_{LF} tend to zero. In this way, we see that for a fixed value of h and k the modified equation is a smoothed version of $u_t + au_x = 0$ and as $h \rightarrow 0$ this smoothing vanishes.

Those of you who have some familiarity with PDE see how this could lead to an existence result: 1) prove existence for the regularized equation and understand the properties of its (unique) solution and 2) take the regularization to zero and use a compactness argument to prove that there's a sequence of solutions of the regularized equation that converge to a solution

of the hyperbolic equation. Of course, this is all irrelevant in this particular case because the hyperbolic equation in question $u_t + \alpha u_x = 0$ not only has explicit solutions but they're unique. However, in general proving existence and/or uniqueness for hyperbolic problems is much Much **MUCH** harder than for parabolic or elliptic problems. The very first proof for a fairly simple nonlinear hyperbolic problem was actually done by proving that a particular numerical method converged. (Glimm, Communications in Pure and Applied Mathematics 18(1965)697-715.)

Finally, note that for $-1 < \mu < 1$

$$D_{\text{EU}} + \frac{h^2}{2k} (1 - \mu) = D_{\text{LF}} \implies D_{\text{EU}} < D_{\text{LF}}.$$

Thus the diffusivity is greater in the modified equation for Lax-Friedrichs than for the modified equation for the explicit upwinding scheme. This explains what you observed in Figure 1 of the February 5 notes: the shock was smoothed out more by the Lax-Friedrichs scheme than by the explicit upwind scheme.