Noncharacteristic boundary conditions

In the book, the boundary is locally straightened and the noncharacteristic boundary condition is given in (35) on page 105. However, the noncharacteristic boundary condition can also be given for the unflattened boundary; see (36) on page 106. Please reconcile (36) with (35).

Inviscid Burger's Equation

Consider $u_{x_2} + uu_{x_1} = 0$ on $U = \{x_2 > 0\} \subset \mathbb{R}^2$ with C^1 boundary data $u(x_0, 0) = g(x_0)$.

a) What are the characteristic ODE? What is the initial data?

b) Solve the characteristic ODE, resulting in explicit solutions $p_1(s)$, $p_2(s)$, $z(s)$, $x_1(s)$, and $x_2(s)$.

c) Assume $g(x) = \tanh(x)$. What are the explicit solutions? Plot the characteristics $(x_1(s), x_2(s))$ in the (x_1, x_2) plane. Given $(X_1, X_2) \in U$ can you find x_0 and $\tilde{s}$ such that $(X_1, X_2) = (x_1(\tilde{s}), x_2(\tilde{s}))$? What is $u(X_1, X_2) = z(x_1(\tilde{s}), x_2(\tilde{s}))$? Graph a few "snapshots" of the solution by graphing $u(X_1, X_2)$ for fixed values of X_2.

d) Assume $g(x) = -\tanh(x)$. Repeat the above. What went wrong? Could you have predicted that something would go wrong by looking at the solutions $(p_1(s), p_2(s))$? In terms of the solution $u(X_1, X_2)$ what does this reflect?

The Eikonal Equation on the interval

Consider the eikonal equation $|u_x| = 1$ on the interval $(0, 1)$ with boundary data $u(0) = u(1) = 0$.

a) What are the characteristic ODE? What is the initial data? (NOTE: at each point on the boundary you have two possible sets of initial data.)

b) Take the initial data $p(0) = 1$ at the left-hand boundary point and $p(0) = -1$ at the right-hand boundary point. Explicitly solve the characteristic ODE. Now, given $X \in (0, 1)$, note that you cannot uniquely determine x_0 on the boundary and $\tilde{s}$ so that $x(\tilde{s}) = X$. So let's make some choices.... What solution $u(X)$ do you construct if you assign X to the characteristic starting at the nearest boundary point? What solution do you construct if you assign X to the characteristic starting at $x_0 = 0$ if $X < \alpha$ and to the characteristic starting at $x_0 = 1$ if $X \geq \alpha$? (Assume $\alpha \in (0, 1)$ is fixed.) Graph your solutions. Is one solution "better" than the others? What does it correspond to geometrically?

c) What types of solutions would you have found if you had chosen the other options for $p(0)$ on the boundary?

d) In b), you constructed infinitely many solutions. All but one of them were discontinuous. Without using the method of characteristics, construct infinitely many continuous solutions.

Interesting fact: the theory of *viscosity solutions* was introduced precisely to select the "desired" solution from the sea of solutions you've just discovered.

The Eikonal Equation on the strip

Consider the eikonal equation $|Du| = 1$ on the strip $U = \{0 < x_1 < 1\} \subset \mathbb{R}^2$ with boundary data $u(0, x_2) = u(1, x_2) = 0$ for all $x_2 \in R$.

Repeat a), b), and d) above.

The Eikonal Equation on the disk

Consider the eikonal equation $|Du| = 1$ on the disk $U = \{x_1^2 + x_2^2 < 1\} \subset \mathbb{R}^2$ with boundary data $u(x_1, x_2) = 0$ for all $(x_1, x_2) \in \partial U$.

a) What are the characteristic ODE? What is the initial data? (NOTE: at each point on the boundary you have four possible sets of initial data.)

b) Take the initial data $(p_1(0), p_2(0))$ to point "inwards". Explicitly solve the characteristic ODE. Now, given $X \in U$, note that you cannot uniquely determine x_0 on the boundary and $\tilde{s}$ so that $(x_1(\tilde{s}), x_2(\tilde{s})) = X$. So let's make some choices.... What solution $u(X)$ do you construct if you assign X to the characteristic starting at the nearest boundary point? What solution do you construct if you assign X to the characteristic starting at $x_0 \in \partial U$ where you require that x_0 lie on the "southern hemisphere"? (That is $(x_0)_2 \leq 0$.) Describe your solutions. Is one solution "better" than the others? What does it correspond to geometrically? The second solution, where in $\overline{U}$ is it discontinuous?

c) Above, you constructed infinitely many solutions. All but one of them were discontinuous. Without using the method of characteristics, construct infinitely many continuous solutions.

The Eikonal equation on other domains

a) What would you expect to get as the "preferred" solution of the Eikonal equation on an ellipse? Where would it fail to be differentiable?

b) What would you expect to get as the "preferred" solution of the Eikonal equation on a rectangle? Where would it fail to be differentiable?

Interesting fact: Level set methods, which are powerful computational methods used in computer vision, image processing, and general scientific computing are built on trying to compute solutions of Eikonal equations. Also, equally difficult equations show up in control theory and in finance. Computing their solutions is *tricky*!