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Sept 29, 2004

We now know how to use the fundamental solution  $\Phi$  to solve

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{in } \mathbb{R}^n \times \{t=0\} \end{cases}$$

How can we use the fundamental solution to solve

$$\textcircled{*} \quad \begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{in } \mathbb{R}^n \times \{t=0\} \end{cases} \quad ?$$

Step 1: If  $v$  solves  $\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{in } \mathbb{R}^n \times \{t=0\} \end{cases}$

and  $w$  solves  $\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = 0 & \text{in } \mathbb{R}^n \times \{t=0\} \end{cases}$

then  $v+w$  solves  $\textcircled{*}$ . We know

$$v(x, t) = \int_{\mathbb{R}^n} \Phi(x-y, t) g(y) dy$$

if  $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ . Let's assume  $g$  is that nice. Then we're down to solving

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = 0 & \text{in } \mathbb{R}^n \times \{t=0\} \end{cases}$$

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To do this, we'll use Duhamel's principle which works very generally, not just for the inhomogeneous heat equation.

Fix  $s > 0$  and let  $u(x, t; s)$  be the solution

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (s, \infty) \\ u = f(x, s) & \text{in } \mathbb{R}^n \times \{t=s\} \end{cases}$$

Now, define

$$u(x, t) := \int_0^t u(x, t; s) ds$$

then

$$u_t(x, t) = u(x, t; t) + \int_0^t u_t(x, t; s) ds$$

$$\Delta u(x, t) = \int_0^t \Delta u(x, t; s) ds$$

$$\begin{aligned} \Rightarrow u_t - \Delta u &= u(x, t; t) + \int_0^t u_t(x, t; s) \Delta u(x, t; s) ds \\ &= u(x, t; t) \quad \text{since } u(x, t; s) \text{ solves } \textcircled{2} \\ &\qquad \text{whenever } t > s \\ &= f(x, t) \quad \text{by } \textcircled{2} \end{aligned}$$

So formally we have a solution of the inhomogeneous heat equation.

$$u(x, t) = \int_0^t u(x, t; s) ds$$

$$= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y, s) dy ds$$

Can we prove that the above really works?

Theorem: Assume  $f \in C_1^2(\mathbb{R}^n \times [0, \infty))$  and  $f$  has compact support in  $\mathbb{R}^n \times [0, \infty)$ . Let

$$u(x, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y, s) dy ds . \text{ Then}$$

$$(i) \quad u \in C_1^2(\mathbb{R}^n \times (0, \infty))$$

$$(ii) \quad u_t - \Delta u = f \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

$$(iii) \quad \lim_{\substack{(x, t) \rightarrow (x_0, 0) \\ t \downarrow 0}} u(x, t) = 0 \quad \text{for each } x_0 \in \mathbb{R}^n$$

Defn:  $f \in C_1^2(\mathbb{R}^n \times (0, \infty))$  if

$$f, \frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j}, \frac{\partial f}{\partial t} \in C(\mathbb{R}^n \times (0, \infty))$$

Note: For the homogeneous heat equation,  $u$  becomes instantaneously  $C^\infty$ . For the inhomogeneous heat equation,  $u$  seems to be no better than  $f$ .

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Proof: To show that  $u_t$  exists, we try  
a finite difference approximation on

$$u(x, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y, s) dy ds$$

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{u(x, t+h) - u(x, t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y, s) dy ds \\ &+ \lim_{h \rightarrow 0} \int_0^t \int_{\mathbb{R}^n} \left( \frac{\Phi(x-y, t+h-s) - \Phi(x-y, t-s)}{h} \right) f(y, s) dy ds \end{aligned}$$

To take the limit inside the second integral,  
we need that the integrand have a pointwise  
limit almost everywhere and that the absolute  
value of the integrand can be bounded by a  
function in  $L^1(\mathbb{R}^n \times (0, t))$ . It's natural to try and  
bound the integrand with something like

$$\|f\|_{L^\infty(\mathbb{R}^n \times (0, t))} |\Phi_t(x-y, t-s)|$$

but such a function is not in  $L^1$  because  
 $\Phi_t$  has a non-integrable singularity at  $s=t$ . So we  
need to write  $u(x, t)$  in such a way as to be  
able to differentiate  $f$  rather than  $\Phi$ .

$$u(x, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) f(x-y, t-s) dy ds$$

since  $f \in C^2_c(\mathbb{R}^n \times [0, \infty))$  and  $f$  has compact support we will be able to show that  $u \in C^2_c(\mathbb{R}^n \times (0, \infty))$  using the same arguments that we used in studying the Poisson problem. i.e. we use  $\Phi \in L^1$  and  $f_t, f_{x_i}, f_{x_i x_j}$  continuous and bounded in order to be able to apply Lebesgue Dominated convergence theorem as needed.

Doing this, we find

$$\begin{aligned} u_t(x, t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) f_t(x-y, t-s) dy ds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y, t) f(x-y, 0) dy \end{aligned}$$

$$\Delta u(x, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) \Delta_x f(x-y, t-s) dy ds$$

We want to see if

$$u_t - \Delta u = f \quad \text{on } \mathbb{R}^n \times (0, \infty).$$

$$\begin{aligned} u_t - \Delta u &= \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) [f_t(x-y, t-s) - \Delta_x f(x-y, t-s)] dy ds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y, t) f(x-y, 0) dy \end{aligned}$$

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We would like to integrate by parts, except that  $\Phi(y, s)$  is singular at  $s=0$ . So we need to isolate  $s=0$ .

$$\begin{aligned}
 M_t - \Delta u &= \int_{\varepsilon}^t \int_{\mathbb{R}^n} \Phi(y, s) \left[ -\frac{\partial}{\partial s} f(x-y, t-s) - \Delta_y f(x-y, t-s) \right] dy ds \\
 &\quad + \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y, s) \left[ -\frac{\partial}{\partial s} f(x-y, t-s) - \Delta_y f(x-y, t-s) \right] dy ds \\
 &\quad + \int_{\mathbb{R}^n} \Phi(y, t) f(x-y, 0) dy \\
 &= I_\varepsilon + J_\varepsilon + K.
 \end{aligned}$$

$$\begin{aligned}
 |J_\varepsilon| &\leq \left( \|f_t\|_{L^\infty(\mathbb{R}^n \times (0, \infty))} + \|\Delta f\|_{L^\infty(\mathbb{R}^n \times (0, \infty))} \right) \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y, s) dy ds \\
 &= \varepsilon \left( \|f_t\|_{L^\infty(\mathbb{R}^n \times (0, \infty))} + \|\Delta f\|_{L^\infty(\mathbb{R}^n \times (0, \infty))} \right)
 \end{aligned}$$

and so  $J_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

$$\begin{aligned}
 I_\varepsilon &= \int_{\varepsilon}^t \int_{\mathbb{R}^n} \left[ \frac{\partial}{\partial s} \Phi(y, s) - \Delta_y \Phi(y, s) \right] f(x-y, t-s) dy ds \\
 &\quad + \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x-y, t-\varepsilon) dy - \int_{\mathbb{R}^n} \Phi(y, t) f(x-y, 0) dy
 \end{aligned}$$

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(Make sure you understand what happened to the boundary terms from the integration by parts in the  $y$  derivatives!)

$$I_\varepsilon = \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x-y, t-\varepsilon) dy - \int_{\mathbb{R}^n} \Phi(y, t) f(x-y, 0) dy$$

$$\text{Since } \Phi_\varepsilon - \Delta \Phi = 0.$$

$$\Rightarrow u_t - \Delta u = I_\varepsilon + \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x-y, t-\varepsilon) dy$$

take limit as  $\varepsilon \rightarrow 0$

$$I_\varepsilon \rightarrow 0 \quad \text{and} \quad \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x-y, t-\varepsilon) dy \rightarrow f(x, t).$$

hence  $u_t - \Delta u = f(x, t)$ , as desired.

We've shown that  $u \in C^2_c(\mathbb{R}^n \times [0, \infty))$  and that  $u$  solves the PDE. It remains to show that  $u$  achieves the initial data:

$$\lim_{\substack{(x,t) \rightarrow (x_0,0) \\ x \in \mathbb{R}^n, t > 0}} u(x, t) = 0$$

$$|u(x, t)| = \left| \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y, s) dy ds \right|$$

$$\leq \int_0^t \int_{\mathbb{R}^n} |\Phi(x-y, t-s)| |f(y, s)| dy ds \leq \|f\|_{L^\infty} \int_0^t ds = t \|f\|_{L^\infty}$$

and so  $\lim_{(x,t) \rightarrow (x_0, 0)} |u(x,t)| \geq \sin \alpha$  since  $\|f\|_{L^\infty} \rightarrow 0$ .

We now have the analogues of the  $R^n$  theorems that we had for Poisson's equation. Is there an analogue of the Mean Value property?

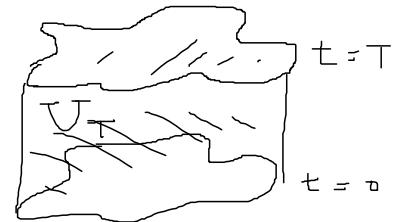
Yes, but not on nice spheres in  $R^n$ . We'll have "heat balls" in  $R^n \times (0, \infty)$ .

Assume  $\Omega$  is open + bounded in  $R^n$  fix  $T > 0$ .

We define the parabolic cylinder

$$\Omega_T = \Omega \times (0, T]$$

the parabolic boundary of  $\Omega_T$  is



$$\text{i.e. } (\Omega \times \{t=0\}) \cup (\partial\Omega \times (0, T])$$

Defn: for fixed  $x \in R^n$   $t \in R$ ,  $r > 0$  we define

$$E(x, t, r) = \left\{ (y, s) \in R^{n+1} \mid s \leq t, \Phi(x-y, t-s) \geq \frac{1}{r^n} \right\}$$

Q: How is this related to balls for  $\Phi$  (the fundamental solution of  $\Delta u = 0$ )?

If  $\underline{\Phi}$  = fundamental solution of  
 $\Delta u = 0$

then level sets of  $\underline{\Phi}$  are spheres. And

$$B(x, r) = \{y \in \mathbb{R}^n \mid \underline{\Phi}(x-y) > C_r\}$$

where  $C_r$  is a function of  $n$  and  $r$ .

$E(x, t; r)$  is like this except you're looking for all  $(y, s)$  such that  $\underline{\Phi}(x-y, t-s) \geq \frac{1}{r^n}$ .

To better understand this, consider

$$\{(y, s) \mid s > 0 \text{ and } \underline{\Phi}(y, s) \geq \frac{1}{r_0^n}\}.$$

i.e. You start w/ a  $\delta$  function at  $x=0$   $t=0$  and ask "what points  $y$  and what times  $s$  have  $\underline{\Phi}(y, s) \geq \frac{1}{r_0^n}$ ?" This will be a radially symmetric set (at any fixed  $s > 0$ ). It will include  $(0, 0)$  and it will include  $(0, t_{\max})$ .

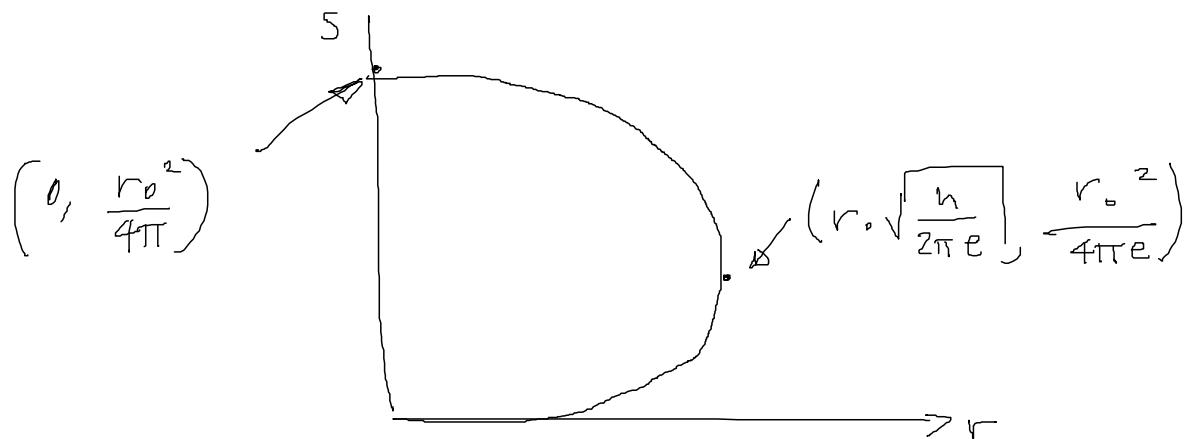
where

$$\underline{\Phi}(0, t_{\max}) = \frac{1}{r_0^n} \Rightarrow t_{\max} = \frac{r_0^n}{4\pi}$$

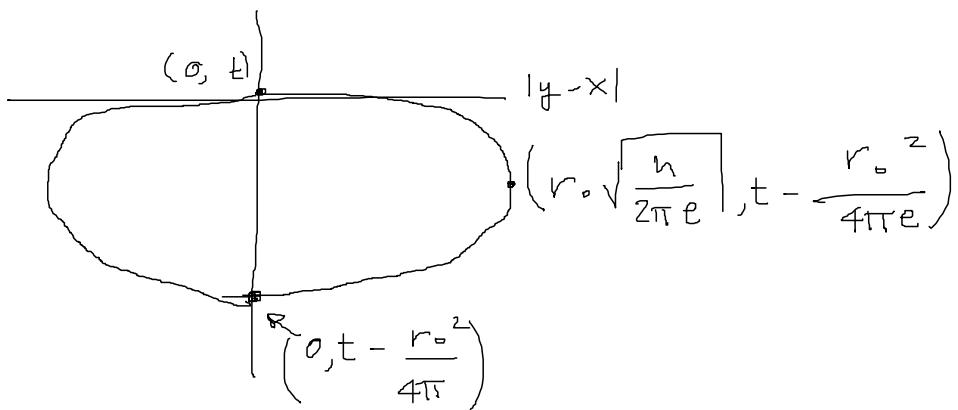
Fix  $s > 0$  and seek  $r$  such that

$$\underline{\Phi}(r, s) \geq -\frac{1}{r_0} r$$

$$\Rightarrow 0 \leq r \leq \sqrt{4s \ln\left(\frac{r_0^h}{(4\pi s)^{h/2}}\right)}$$



Now, to understand  $E(x, t; r)$  we reverse time and center the ball at  $x$ .



theorem: Let  $u \in C_1^2(U_T)$  solve the heat equation  $u_t = \Delta u$ . Then for each  $E(x, t; r) \subseteq U_T$  one has

$$u(x, t) = \frac{1}{4\pi^n} \iint_{E(x, t; r)} u(y, s) \frac{|x-y|^2}{(t-s)^n} dy ds$$

$$E(x, t; r)$$

So  $u(x, t)$  is a weighted average of  $u(y, s)$  over the heat ball  $E(x, t; r)$ . Note that  $E(x, t; r)$  only admits  $u(y, s)$  for  $s < t$ , consistent with causality.

proof: Without loss of generality, take  $x=0$  and  $t=0$ . (heat equation is invariant wrt translation in space and time.)

$$\text{Let } \phi(r) := \frac{1}{r^n} \iint_{E(0, 0; r)} u(y, s) \frac{|y|^2}{s^n} dy ds$$

$$E(0, 0; r)$$

Introduce a change of variables

$$y = r\tilde{y} \quad s = r^2\tilde{s}$$

then  $E(0, 0; r) \rightarrow E(0, 0; 1)$  and

$$\phi(r) = \iint_{E(0, 0; 1)} u(r\tilde{y}, r^2\tilde{s}) \frac{|\tilde{y}|^2}{\tilde{s}^n} d\tilde{y} d\tilde{s}$$

We can now compute  $\frac{d\phi}{dr}$ .

let  $\tilde{y} \rightarrow y$  and  $\tilde{s} \rightarrow s$ .

$$\frac{d\phi}{dr} = \iint \sum_{i=1}^n y_i \frac{\partial u}{\partial y_i}(ry, r^2s) \frac{|y|^2}{s^2} + 2rs \frac{\partial u}{\partial s}(ry, r^2s) \frac{|y|^2}{s^2} dy ds$$

$E(0,0;1)$

$$= \iint \frac{|y|^2}{s^2} \sum_{i=1}^n y_i \frac{\partial u}{\partial y_i}(ry, r^2s) + 2r \frac{|y|^2}{s} \frac{\partial u}{\partial s}(ry, r^2s) dy$$

$E(0,0;1)$

now change variables back

$$\tilde{y} = ry \quad r^2s = \tilde{s}$$

$$\frac{d\phi}{dr} = \frac{1}{r^{n+1}} \iint \frac{|y|^2}{s^2} \sum_{i=1}^n y_i \frac{\partial u}{\partial y_i}(y, s) + 2 \frac{|y|^2}{s} \frac{\partial u}{\partial s}(y, s) dy$$

$E(0,0;r)$

$$= : A + B$$

Our goal is to show that  $A + B = 0$

by definition,  $(y, s) \in E(0,0;r)$

$$\Leftrightarrow \Phi(y, -s) \geq \frac{1}{r^n}$$

$$\Leftrightarrow \ln(\Phi(y, -s)) \geq \ln\left(\frac{1}{r^n}\right)$$

$$\Leftrightarrow \ln(\phi$$

$$\Leftrightarrow \ln(\Phi(y, -s)) - \ln\left(\frac{1}{r^n}\right) \geq 0$$

w/ equality if  $(y, s) \in \partial E(0, 0; r)$

$$\text{let } \Psi(y, s) := \ln(\Phi(y, -s)) - \ln\left(\frac{1}{r^n}\right)$$

$$= -\frac{n}{2} \ln(-4\pi s) + \frac{|y|^2}{4s} + n \ln(r)$$

$$\text{then } \frac{\partial \Psi}{\partial y_i} = \frac{y_i}{2s} \Rightarrow \sum_{i=1}^n y_i \frac{\partial \Psi}{\partial y_i} = \frac{|y|^2}{2s}$$

$$B = \frac{1}{r^{n+1}} \iint_{E(0,0; r)} 2 \frac{|y|^2}{s} \frac{\partial u}{\partial s}(y, s) dy ds$$

$$= \frac{4}{r^{n+1}} \iint_{E(0,0; r)} \frac{\partial u}{\partial s}(y, s) \sum_{i=1}^n y_i \frac{\partial \Psi}{\partial y_i}(y, s) dy ds$$

I do integration by parts for each  $y_i$  derivative. The boundary terms are

$$\frac{\partial u}{\partial s}(y, s) y_i \Psi(y, s) \text{ which}$$

vans. since  $\Psi(y, s) = 0$  for  $(y, s) \in \partial E(0, 0; r)$

$$B = -\frac{4}{r^{n+1}} \iint_{E(0,0; r)} \Psi(y, s) \sum_{i=1}^n \left( y_i \frac{\partial u}{\partial y_i}(y, s) \right) dy ds$$

$$= -\frac{4}{r^{n+1}} \iint_{E(0,0; r)} \Psi(y, s) \sum_{i=1}^n \left[ y_i \frac{\partial^2 u}{\partial y_i \partial s} + \frac{\partial u}{\partial s} \right] dy ds$$

$$= -\frac{4}{r^{n+1}} \iint_{E(0,0; r)} n \Psi(y, s) \frac{\partial u}{\partial s}(y, s) + \Psi(y, s) \sum_{i=1}^n y_i \frac{\partial^2 u}{\partial y_i \partial s}(y, s) dy ds$$

Integrate by parts with respect to  $s$  on  
the second term

$$= -\frac{4}{r^{n+1}} \iint n \Psi(y, s) \frac{\partial u}{\partial s}(y, s) dy ds + \frac{4}{r^{n+1}} \iint \frac{\partial \Psi}{\partial s}(y, s) \sum_{i=1}^n y_i \frac{\partial u}{\partial y_i} dy ds$$

$$\text{Now } \frac{\partial \Psi}{\partial s} = -\frac{n}{2} \frac{1}{s} - \frac{|y|^2}{4s^2} \quad S_1$$

$$B = -\frac{4}{r^{n+1}} \iint n \Psi \frac{\partial u}{\partial s} - \frac{2n}{r^{n+1}} \iint \frac{1}{s} \sum_{i=1}^n y_i \frac{\partial u}{\partial y_i} - \frac{1}{r^{n+1}} \iint \frac{|y|^2}{s^2} \sum_{i=1}^n y_i \frac{\partial u}{\partial y_i}$$

$$= -\frac{4}{r^{n+1}} \iint n \Psi \frac{\partial u}{\partial s} - \frac{2n}{r^{n+1}} \iint \frac{1}{s} \sum_{i=1}^n y_i \frac{\partial u}{\partial y_i} - A$$

hence

$$\frac{d\phi}{dr} = A + B = -\frac{4n}{r^{n+1}} \iint \Psi \frac{\partial u}{\partial S} - \frac{2n}{r^{n+1}} \iint \frac{1}{S} \sum_{i=1}^n y_i \frac{\partial u}{\partial y_i}$$

recall that  $u$  is a solution of the heat equation. Hence

$$\frac{\partial u}{\partial S} = \Delta u$$

$$\frac{d\phi}{dr} = -\frac{4n}{r^{n+1}} \iint \Psi(y, s) \Delta_y u(y, s) dy ds - \frac{2n}{r^{n+1}} \iint \frac{1}{S} \sum_{i=1}^n y_i \frac{\partial u}{\partial y_i}$$

Now use the divergence theorem  
and  $\Psi = 0$  on  $\partial E(r_0, r; r)$

$$\frac{d\phi}{dr} = \frac{4n}{r^{n+1}} \iint \nabla \Psi \cdot \nabla u dy ds - \frac{2n}{r^{n+1}} \iint \frac{1}{S} \sum_{i=1}^n y_i \frac{\partial u}{\partial y_i}$$

since  $\frac{\partial \Psi}{\partial y_i} = \frac{y_i}{2S}$ , the two terms cancel,

therefore  $\frac{d\phi}{dr} = 0$  for  $r \in (0, R)$  where  
 $R$  is the largest  $R$   
such that

$$E(x, t; R) \subseteq U_T$$

It follows that

$$\phi(r) = \lim_{r \rightarrow 0} \phi(r)$$

$$= \lim_{r \rightarrow 0} \left[ \frac{1}{r^n} \iint u(y, s) \frac{|y|^2}{s^2} dy ds \right]$$

$$= u(x, t) \lim_{r \rightarrow 0} \left[ \frac{1}{r^n} \iint_{E(0, 0; r)} \frac{|y|^2}{s^2} dy ds \right]$$

$$= 4u(x, t),$$

This shows that

$$u(x, t) = \frac{1}{4r^n} \iint_{E(x, t; r)} u(y, s) \frac{|x-y|^2}{(t-s)^2} dy ds$$

for any ball  $E(x, t; r) \subseteq U_T$ .

So why does that integral equal 4? Do this explicitly

$$\frac{1}{r_0^n} \iint_{E(0, 0; r_0)} \frac{|y|^2}{s^2} dy ds = \frac{1}{r_0^n} \int_{-\frac{r_0^2}{4\pi}}^0 ds \int_0^{r_{\max}(s)} \frac{r^2}{s^2} nr^{n-1} \omega(n) dr$$

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where

$$V_{\max}(s) = \sqrt{2ns \ln\left(\frac{-4\pi s}{r_0^2}\right)}$$

i.e. we have the integral

$$\begin{aligned} & \frac{1}{r_0^n} \frac{n\alpha(n)}{n+2} \int_{-\frac{r_0^2}{4\pi}}^0 \frac{1}{s^2} (V_{\max}(s))^{n+2} ds \\ &= \frac{1}{r_0^n} \frac{n\alpha(n)}{n+2} \int_{-\frac{r_0^2}{4\pi}}^0 (2n)^{\frac{n+2}{2}} s^{\frac{n-2}{2}} \left[ \ln\left(\frac{-4\pi s}{r_0^2}\right) \right]^{\frac{n+2}{2}} ds \\ &= 4. \end{aligned}$$

Confession: I used maple to verify the integral for some small even numbers and believe I can do it by hand if needed. Would welcome some python start to show me a clever way to do it for odd n.