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We'll now study the heat equation

$$u_t = \Delta u \quad \text{in } \mathbb{T}$$

and the nonhomogeneous heat equation

$$u_t = \Delta u + f \quad \text{in } \mathbb{T}$$

where f is a specified real valued function

$$f: \mathbb{T} \times [0, \infty) \rightarrow \mathbb{R}$$

The unknown u is defined on $\overline{\mathbb{T}} \times [0, \infty)$

where $\mathbb{T} \subseteq \mathbb{R}^n$ is open

To study u , we again seek exact solutions.
Since the Δ is rotation invariant, seeking
a radially symmetric solution is natural.

Another natural solution is the self-similar
solution

$$u(x, t) := \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)$$

If $\text{supp}(v) = B(0, 1)$ then at each time t ,

$\text{supp}(u) = B(0, t^\beta)$. For heat, we'd then expect
 $\beta > 0$ so that the support expands in time.

Similarly, at any time t ,

$$\max(u(\cdot, t)) = \frac{1}{t^\alpha} \max(v)$$

so we expect $\alpha > 0$. Let's look for a self-similar solution of

$$u_t = \Delta u \quad \text{in } \mathbb{R}^n$$

$$\Rightarrow -\alpha \frac{1}{t^{\alpha+1}} v\left(\frac{x}{t^\beta}\right) - \beta \frac{1}{t^\alpha} \frac{x}{t^{\beta+1}} \cdot D_y v\left(\frac{x}{t^\beta}\right) = \frac{1}{t^{\alpha+2\beta}} \Delta_y v\left(\frac{x}{t^\beta}\right)$$

$$\text{where } y = \frac{x}{t^\beta}$$

$$\Rightarrow -\frac{\alpha}{t^{\alpha+1}} v(y) - \frac{\beta}{t^{\alpha+1}} y \cdot D_y v(y) = \frac{1}{t^{\alpha+2\beta}} \Delta_y v(y)$$

want the time dependence to cancel,
removing the time-derivatives from the PDE

$$\Rightarrow \alpha + 1 = \alpha + 2\beta \Rightarrow \beta = \frac{1}{2}$$

$$\Rightarrow -\alpha v - \frac{1}{2} y \cdot D_y v = \Delta_y v$$

How do we determine α ?? Mass conservation.

$\int_{\mathbb{R}^n} u(x, t) dx$ should be independent of t

$$\int_{\mathbb{R}^n} u(x, t) dx = \int_{\mathbb{R}^n} \frac{1}{t^\alpha} v\left(\frac{x}{\sqrt{t}}\right) dx$$

$$y = \frac{x}{\sqrt{t}} \Rightarrow dy = \frac{1}{t^{n/2}} dx$$

$$= \frac{1}{t^\alpha} t^{n/2} \int_{\mathbb{R}^n} v(y) dy = \frac{t^{n/2}}{t^\alpha} \int_{\mathbb{R}^n} v(y) dy$$

$$\Rightarrow \boxed{\alpha = n/2}$$

So we seek $u(x, t) = \frac{1}{t^{n/2}} v\left(\frac{x}{\sqrt{t}}\right)$

and the PDE for u has been reduced to a PDE for v :

$$-\frac{n}{2} v - \frac{1}{2} y \cdot Dv = \Delta v$$

Now, since Δ is rotationally invariant, seek radially symmetric solution

$$v(y) = w(|y|) = w(r)$$

$$\Rightarrow \frac{n}{2} w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0$$

PDE reduced to ODE 

$$\frac{n}{2} r^{n-1} w + \frac{1}{2} r^n w' + r^{n-1} w'' + (n-1) r^{n-2} w' = 0$$

$$\frac{1}{2} (r^n w)' + (r^{n-1} w')' = 0$$

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We've written the ODE as an exact derivative so we'll be able to integrate it up. (Note: if a hadn't been chosen to conserve mass of $u(x,t)$ then we wouldn't have found an ODE for w that could be integrated exactly.)

$$\Rightarrow r^{n-1}w' + \frac{1}{2}r^n w = a \quad \textcircled{D}$$

for some constant a . Let's evaluate a somehow.

Idea 1: Assume u is smooth at $x=0 \Rightarrow u$ smooth at $y=0 \Rightarrow w'(0)=0$. Then evaluate the ODE \textcircled{D} at $r=0 \Rightarrow a=0$

Idea 2: Assume u has compact support or vanishes really fast. faster than $|x|^n$. And Du vanishes faster than $|x|^{n-1}$. Then by evaluating \textcircled{D} at larger & large values of r , we find $a=0$.

No matter what the case, if we find a w that satisfies

$$r^{n-1}w' + \frac{1}{2}r^n w = 0$$

then we've found a solution. There might be others, for $a \neq 0$.

$$\Rightarrow r^{n-1} w' = -\frac{1}{2} r^n w$$

$$\Rightarrow \cancel{r} w' = -\frac{1}{2} r w \quad \text{wherever } r > 0$$

$$\Rightarrow w(r) = b e^{-r^2/4}$$

Now, we'll choose b such that w has area 1:

$$\begin{aligned} \int_{\mathbb{R}^n} b e^{-|y|^2/4} dy &= b \int_{\mathbb{R}^n} e^{-|y|^2/4} dy \\ &\stackrel{z = y/2}{=} 2^n b \int_{\mathbb{R}^n} e^{-|z|^2} dz \quad dz = \frac{1}{2^n} dy \\ &= 2^n b (\sqrt{\pi})^n = 1 \quad \Rightarrow \quad b = \left(\frac{1}{\sqrt{4\pi}}\right)^n \end{aligned}$$

$$\Rightarrow w(r) = \frac{1}{(4\pi)^{n/2}} e^{-r^2/4}$$

$$\Rightarrow v(y) = \frac{1}{(4\pi)^{n/2}} e^{-|y|^2/4}$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}}$$

is a self-similar solution of $ut = \Delta u$

Note that if you fix $x \neq 0$ and take $t \rightarrow 0$, $u(x,t) \rightarrow 0$.

Definition the function

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{1/2}} e^{-|x|^2/4t} & t > 0, x \in \mathbb{R}^n \\ 0 & t < 0, x \in \mathbb{R}^n \end{cases}$$

is called the fundamental solution of
the heat equation.

Consider the initial value problem (or Cauchy problem)

$$\begin{cases} u_t = \Delta u & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{in } \mathbb{R}^n \times \{t=0\} \end{cases}$$

since $(x,t) \rightarrow \Phi(x,t)$ solves the heat equation away from $t=0$

and $(x,t) \rightarrow \Phi(x-y, t)$ for fixed y also solves the heat equation, we try

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y, t) g(y) dy$$

if our intuition (i.e. desperation) is right,

then $\Phi(x,t) \rightarrow \delta_0(x)$ as $t \rightarrow 0$ and so the

RHS would go to $g(x)$.

Theorem: Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and let

$$u(x, t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} g(y) dy & t > 0 \\ g(x) & t = 0 \end{cases}$$

Then

$$(i) \quad u \in C^\infty(\mathbb{R}^n \times (0, \infty))$$

$$(ii) \quad u_t = \Delta u \text{ in } \mathbb{R}^n \times (0, \infty)$$

$$(iii) \quad \lim_{\substack{(x,t) \rightarrow (x_0, 0) \\ x \in \mathbb{R}^n, t > 0}} u(x, t) = g(x_0) \quad \text{for each } x_0 \in \mathbb{R}^n$$

Note 1: we don't have as tight a constraint on g as we did on f when solving $-\Delta u = f$.

Recall that $f \in C_c^2(\mathbb{R}^n)$ in order for

$$u(x) = \int \Phi(x-y) f(y) dy \quad (\text{different } \Phi)$$

to make sense. Why? the fundamental solution for Laplace's equation had infinite L^1 norm while the fundamental solution for the heat equation has finite norm.

Note 2: we don't assume g has derivatives, unlike f for Poisson's problem.

Note 3: even though g is irregular, u is C^∞ in $x \notin t$!

Proof:

(i) Show that $u(x, t) = \int \Phi(x-y, t) g(y) dy$ is C^∞ in $x \in \mathbb{R}^n$. First, show u is continuous.

Fix $t > 0$

$$\begin{aligned} |u(x, t) - u(z, t)| &= \left| \int_{\mathbb{R}^n} (\Phi(x-y, t) - \Phi(z-y, t)) g(y) dy \right| \\ &\leq \int_{\mathbb{R}^n} |\Phi(x-y, t) - \Phi(z-y, t)| |g(y)| dy \\ &\leq \|g\|_{L^\infty} \int_{\mathbb{R}^n} |\Phi(x-y, t) - \Phi(z-y, t)| dy \end{aligned}$$

We now argue that $\exists \delta > 0$ such that $|x-z| < \delta$

$$\Rightarrow \int_{\mathbb{R}^n} |\Phi(x-y, t) - \Phi(z-y, t)| dy < \varepsilon / \|g\|_{L^\infty}.$$

This requires a little care.

$$\begin{aligned} \int_{\mathbb{R}^n} |\Phi(x-y, t) - \Phi(z-y, t)| dy &= \int_{B(x, R)} |\dots| dy + \int_{\mathbb{R}^n - B(x, R)} |\dots| dy \\ &=: I + J. \end{aligned}$$

$$\begin{aligned} J &\leq \int_{\mathbb{R}^n - B(x, R)} |\Phi(x-y, t)| dy + \int_{\mathbb{R}^n - B(z, R)} |\Phi(z-y, t)| dy \quad \text{Assume } |x-z| < 1 \\ &\leq \int_{\mathbb{R}^n - B(x, R-1)} |\Phi(x-y, t)| dy + \int_{\mathbb{R}^n - B(z, R-1)} |\Phi(z-y, t)| dz \end{aligned}$$

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choose R so large that

$$\int_{\mathbb{R}^n - B(x, R-1)} |\Phi(x-y, t)| dy < \frac{1}{3} \frac{\varepsilon}{\|g\|_{L^\infty}}.$$

for this choice of R , we know $\mathcal{J} < \frac{2}{3} \frac{\varepsilon}{\|g\|_{L^\infty}}$.

Now to show $I < \frac{1}{3} \frac{\varepsilon}{\|g\|_{L^\infty}}$. This will require taking z close to x .

$$I = \int_{B(x, R)} |\Phi(x-y, t) - \Phi(z-y, t)| dy$$

$$= \int_{B(x, R)} |D\Phi(\xi_y, t) \cdot (x-z)| dy \quad \text{where } \xi_y \text{ is some point between } x-y \text{ and } z-y$$

$$\leq \|D\Phi(\cdot, t)\|_{L^\infty} |x-z| \int_{B(x, R)} dy$$

$$= \|D\Phi(\cdot, t)\|_{L^\infty} |x-z| \propto (n) R^n,$$

By choosing $|x-z|$ sufficiently small ($|x-z| < \tilde{\delta}$)

$$\text{then } I < \frac{\varepsilon}{3} \frac{1}{\|g\|_{L^\infty}} \Rightarrow |x-z| < \min\{1, \tilde{\delta}\}$$

$$\Rightarrow |u(x, t) - u(z, t)| < \varepsilon. \checkmark$$

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Similarly, you show that u is continuous in time. To show that u_{x_i} and u_t exist at (x, t) when $t > 0$, you do as expected:

$$\lim_{h \rightarrow 0} u(x, t+h) - u(x, t) = \lim_{h \rightarrow 0} \int_{\mathbb{R}^n} \left[\frac{\Phi(x-y, t+h) - \Phi(x-y, t)}{h} \right] g(y) dy,$$

We have a pointwise limit in the integrand and the integrand is bounded above by an integrable function. So the limit can be taken inside the integral by the Lebesgue Dominated Convergence Theorem.

$$u_t(x, t) = \int_{\mathbb{R}^n} \frac{\partial \Phi}{\partial t}(x-y, t) g(y) dy$$

Argue the other derivatives analogously. Also argue the derivatives are continuous, analogously. Note: be aware that you're using that $e^{-|x|^2}$ beats any polynomial in x !

ii) Now that we know $u \in C^\infty(\mathbb{R}^n, (0, \infty))$ and we have formulas for the derivatives,

$$u_t - \Delta u = \int_{\mathbb{R}^n} [\Phi_t(x-y, t) - \Delta_x \Phi(x-y, t)] g(y) dy = 0$$

So u is a solution, since Φ is.

(iii) Now to show that $u(x,t)$ achieves the initial data g .

i.e. $u \in C(\mathbb{R}^n \times [0, \infty)) \cap C^\infty(\mathbb{R}^n \times (0, \infty))$
and $u(x, 0) = g(x)$.

The argument will be very similar to the argument
for $\begin{cases} \Delta u = f \text{ in } \mathcal{V} & \text{when } \mathcal{V} = \mathbb{R}_+^n \text{ or disc} \\ u = g \text{ on } \partial \mathcal{V} & \text{and you have Green's functions.} \end{cases}$

Fix $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$. Choose $\delta > 0$ so that

$$|y - x_0| < \delta \Rightarrow |g(y) - g(x_0)| < \varepsilon.$$

$$\begin{aligned} \text{now } |u(x, t) - g(x_0)| &= \left| \int \Phi(x-y, t) g(y) dy - \int \Phi(x-y, t) g(x_0) dy \right| \\ &= \left| \int \Phi(x-y, t) (g(y) - g(x_0)) dy \right| \\ &\leq \int_{B(x_0, \delta)} |\Phi(x-y, t)| |g(y) - g(x_0)| dy \\ &\quad + \int_{\mathbb{R}^n - B(x_0, \delta)} |\Phi(x-y, t)| |g(y) - g(x_0)| dy \\ &= I + J. \end{aligned}$$

$$I \leq \int_{B(x_0, \delta)} \Phi(x-y, t) \varepsilon < \varepsilon \quad \text{since } |y - x_0| < \delta.$$

Assume $|x - x_0| < \frac{\delta}{2}$ and $|y - x_0| \geq \delta$

Then $|y - x_0| \leq |y - x| + |x - x_0| < |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x_0|$

$$\Rightarrow \frac{1}{2}|y - x_0| < |y - x|$$

$$\begin{aligned} J &= \int_{\mathbb{R}^n - B(x_0, \delta)} \Phi(x-y, t) (g(y) - g(x_0)) dy \\ &\leq 2\|g\|_{L^\infty} \int_{\mathbb{R}^n - B(x_0)} \Phi(x-y, t) dy \\ &= \frac{2\|g\|_{L^\infty}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n - B(x_0, \delta)} e^{-\frac{|x-y|^2}{4t}} dy \\ &< \frac{2\|g\|_{L^\infty}}{(4\pi t)^{n/2}} \int_{B(x_0, \delta)} e^{-\frac{|y-x_0|^2}{16t}} dy \\ &= \frac{2\|g\|_{L^\infty}}{(4\pi t)^{n/2}} \int_0^\infty e^{-\frac{r^2}{16t}} n \omega(n) r^{n-1} dr \\ &= \frac{2\|g\|_{L^\infty}}{(4\pi t)^{n/2}} t^{\frac{n-1}{2}} \int_{\delta/\sqrt{t}}^\infty e^{-\frac{\rho^2}{16t}} \rho^{n-1} d\rho \\ &< \varepsilon \text{ for } t \text{ sufficiently small.} \end{aligned}$$

This shows that

$$|u(x, t) - g(x_0)| \leq I + J < \varepsilon + \varepsilon \quad \text{if}$$

$$|x - x_0| < \frac{\delta}{2} \quad \text{and}$$

t is small enough.

i.e.,

$$\lim_{\substack{(x,t) \rightarrow (x_0, 0) \\ x \in \mathbb{R}^n, t > 0}} u(x, t) = g(x_0).$$



Note that in the bounding of J , we used that

$$\Phi(x, t) \rightarrow 0 \quad \text{as } t \rightarrow 0 \quad \text{if } x \neq 0.$$

Since x was bounded away from 0, we were essentially using uniform convergence.

Note: If $g \geq 0$ and g is bounded & continuous
then $u(x, t) = \int \Phi(x-y, t) g(y) dy \geq 0$
at all $x \in \mathbb{R}^n$ if $t > 0$. This tells you that
even if g has compact support, instantaneously
the support of u is all of \mathbb{R}^n . i.e. the
heat equation has infinite speed of propagation: if
 $g = 0$ everywhere except on $B(0, 1)$ every point in \mathbb{R}^n
will instantly be influenced by the initial data in $B(0, 1)$.

Q: What if I hadn't known to seek a self-similar solution? Could I have found $\Phi(x, t)$ in some other way?

A: Yes, if you're willing to accept Fourier transforms.

$$\text{Assume } u(x, t) = \int_{\mathbb{R}^n} \hat{u}(\xi, t) e^{i\xi \cdot x} d\xi$$

$$\text{then } u_t = \int_{\mathbb{R}^n} \frac{d}{dt} \hat{u}(\xi, t) e^{i\xi \cdot x} d\xi$$

$$\Delta u = \int_{\mathbb{R}^n} -|\xi|^2 \hat{u}(\xi, t) e^{i\xi \cdot x} d\xi$$

so we need to solve

$$\begin{cases} \frac{d}{dt} \hat{u}(\xi, t) = -|\xi|^2 \hat{u}(\xi, t) \\ \hat{u}(\xi, 0) = \hat{g}(\xi) \end{cases}$$

for each $\xi \in \mathbb{R}^n$.

$$\Rightarrow \hat{u}(\xi, t) = \hat{g}(\xi) e^{-|\xi|^2 t}$$

$$\Rightarrow u(x,t) = \int_{\mathbb{R}^n} \hat{g}(z) e^{-|z|^2 t} e^{iz \cdot x} dz.$$

From how convolutions + Fourier transforms work,

$$u(x,t) = \int_{\mathbb{R}^n} \hat{\Phi}(x-y, t) g(y) dy$$

where $\hat{\Phi}(z, t) = e^{-|z|^2 t}$

since $\hat{\Phi}(x, t) = \int \hat{\Phi}(z, t) e^{iz \cdot x} dz$

$$= \int e^{-|z|^2 t + iz \cdot x} dz$$

$$= \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$$

complete the square in the exponent

and so

$$\hat{\Phi}(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \text{ as desired.}$$