

From last time, we found a solution of

$$-\Delta u = f \text{ in } \mathbb{R}^n$$

where $f \in C_c^2(\mathbb{R}^n)$. Specifically,

$$u(x) = \int_{\mathbb{R}^n} \Phi(y) f(x-y) dy \in C^2(\mathbb{R}^n) \quad \text{and} \quad -\Delta u = f.$$

We will now look at properties of solutions, assuming they exist.

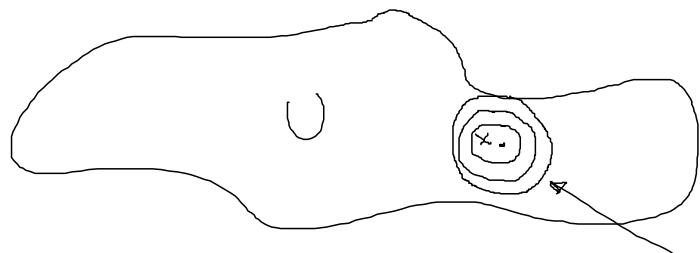
Theorem: Assume $U \subseteq \mathbb{R}^n$ is an open set and assume $u \in C^2(U)$ satisfies $\Delta u = 0$

then

$$u(x) = \int_{\partial B(x,r)} u ds = \int_{B(x,r)} u dy$$

for each ball $B(x,r) \subseteq U$.

This tells us that if you have a harmonic function then its value at a point equals its average value in a ball centred at the point and it also equals its average value over a sphere centred at the point.



$u(x) = \text{average value over the sphere}$

The theorem is very general so either we will have to prove it using only the PDE or we will have to construct the solution u somehow and then use some aspect of the construction. Fortunately, we do the former.

Proof:

$$\text{Let } \phi(r) := \int_{\partial B(x,r)} u(y) dS(y) = \int_{\partial B(0,1)} u(x+rz) dS(z)$$

$$\Rightarrow \frac{d\phi}{dr} = \int_{\partial B(0,1)} Du(x+rz) \cdot \mathbf{z} dS(z) \quad \leftarrow \text{ legit since } u \in C^1.$$

$$= \int_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r} dS(y) \quad \text{and } \frac{y-x}{r} = \mathbf{v}$$

$$= \int_{\partial B(x,r)} \frac{\partial u}{\partial \mathbf{v}}(y) dS(y) \quad = \text{unit outward normal}$$

$$= \frac{1}{n \alpha(n) r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \mathbf{v}}(y) dS(y) = \frac{1}{n \alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y) dy$$

$$= 0 \quad \text{since } \Delta u \equiv 0 \text{ in } V \\ \text{hence in } B(x,r) \text{ since } B(x,r) \subseteq V.$$

This shows that

$$\frac{d\phi}{dr} = 0 \quad \text{and} \quad \phi \text{ is constant as a function of } r$$

further,

$$\phi(r) = \lim_{r \rightarrow 0} \phi(r) = \lim_{r \rightarrow 0} \int_{\partial B(x,r)} u(y) dS(y) = u(x).$$

This shows that

$$u(x) = \int_{\partial B(x,r)} u(y) dS(y) \quad \text{for all balls } B(x,r) \text{ contained in } U.$$

For the average over balls, we derive this from the average over spheres:

$$\int_{B(x,r)} u(y) dy = \int_0^r \left(\int_{\partial B(x,\tilde{r})} u(y) dS(y) \right) d\tilde{r}$$

$$= \int_0^r u(x) n \alpha(n) \tilde{r}^{n-1} d\tilde{r} \quad \leftarrow \begin{matrix} \text{using} \\ \text{the average} \\ \text{over spheres} \end{matrix}$$

$$= u(x) n \alpha(n) \int_0^r \tilde{r}^{n-1} d\tilde{r}$$

$$\propto u(x) \alpha(n) r^n$$

$$\Rightarrow u(x) = \int_{B(x,r)} u(y) dy \quad \text{as desired.} \quad //$$

(4)

Note: You really should sit down and convince yourself that

$$\lim_{r \rightarrow 0} \frac{1}{\#B(x,r)} \int_{B(x,r)} u(y) dS(y) = u(x)$$

or that

$$\lim_{r \rightarrow 0} \left[\int_{B(x,r)} u(y) dS(y) - n \omega(n) r^{n-1} u(x) \right] = 0$$

or that

$$\lim_{r \rightarrow 0} \int_{B(x,r)} u(y) - u(x) dS(y) = 0$$

→ We've shown that any C^2 harmonic function satisfies the mean value property. Similarly, any C^2 function that satisfies the MV property must be harmonic!

Thm: if $u \in C^2(U)$ satisfies

$$u(x) = \frac{1}{\#B(x,r)} \int_{B(x,r)} u(y) dS(y)$$

for each ball $B(x,r) \subseteq U$, then u must be harmonic.

Proof: Assume not. Assume $u \neq 0$ in Ω . Since u is continuous, this means $\exists x \in U$ such

that $\Delta u > 0$ in a neighborhood of x . (or $\Delta u < 0$ in a neighborhood of x . WLOG, assume $\Delta u > 0$ in a neighborhood of x .)

\Rightarrow if $\phi(r) := \int_{\partial B(x,r)} u(y) dS(y)$ then

for r sufficiently small,

$$\frac{r}{n} \int_{B(x,r)} \Delta u(y) dy > 0$$

But this means that $\frac{d\phi}{dr} > 0$ for r sufficiently small. This contradicts that $\phi(r)$ is constant, equalling x

The mean-value properties give us some very useful, physically intuitive properties: the maximum principles.

thm: Suppose $u \in C^2(\bar{U}) \cap C(\bar{U})$ is harmonic in U .

Assume U is open and bounded.

then 1) $\max_{\bar{U}} u = \max_{\partial U} u$

2) if U is connected and $\exists x_0 \in U$ such that

$$u(x_0) = \max_{\bar{U}} u \text{ then } u \text{ is constant.}$$

(b)

Note: i) is the "maximum principle"
ii) is the "strong maximum principle".

2) says that the only way an interior point can be a maximum point is if the function u is constant.

This is obvious for $U \subseteq \mathbb{R}$ since

$u_{xx} = 0 \Rightarrow u(x) = mx + b$ and if the maximum is achieved at an interior point then $m=0$ and $u \equiv b$.

Proof: Will prove 2) and 1) will follow.

Assume $x_0 \in U$ and $u(x_0) = M = \max_{\bar{U}} u$

for any $0 < r < \text{dist}(x_0, \partial U)$ we know

$$M = u(x_0) = \int_{B(x_0, r)} u(y) dy \leq M$$

If $\exists y \in B(x_0, r)$ such that $u(y) < M$ then by continuity, $u < M$ in a neighborhood of y and the $\int_{B(x_0, r)} u(y) dy < M$. Which would yield a

contradiction. $\Rightarrow u \equiv M$ in $B(x_0, r)$.

This proves $\{x \in U \mid u(x) = M\}$ is an open subset of U . On the other hand, this set is relatively closed in U .

(7)

Since U is connected then this forces

$$\{x \in U \mid u(x) = M\} \text{ to be all of } U,$$

$\Rightarrow u$ is constant in U .

Now we use 2) to prove 1).

$$\text{Certainly, } \max_{\overline{U}} u \geq \max_{\partial U} u \quad \text{since } \partial U \subseteq \overline{U}.$$

Assume $\max_{\overline{U}} u > \max_{\partial U} u$. Then the maximum is

achieved at some interior point x_0 . Let U_{x_0} be the connected component containing x_0 . By 2), we know that $u = \text{constant} = u(x_0)$ in U_{x_0} .

Since $u \in C(\overline{U})$, we know that $\lim_{\substack{x \rightarrow \partial U_{x_0} \\ y \in U}} u(x) = u(x_0) = u(y)$

Contradicting $\max_{\partial U} u < \max_{\overline{U}} u$ //

Note that $u \in C^2(U)$ & u harmonic & U connected yields $u(x_0) = \max_U u \Rightarrow u \equiv \text{constant in } U$.

We needed $u \in C(\overline{U})$ when we wanted to say that the maximum was achieved on ∂U .

(8)

Cor: Assume U is open and bounded.

If U is connected and $u \in C^2(U) \cap C(\bar{U})$

and $\begin{cases} \Delta u = 0 \text{ in } U \\ u = g \text{ on } \partial U \end{cases}$

and $g \geq 0$ then if $g > 0$ at some point on ∂U
then $u > 0$ in U .

Cor: Assume U is open and bounded.

assume $g \in C(\partial U)$, $f \in C(U)$ then

there exists at most one solution $u \in C^2(U) \cap C(\bar{U})$

of $\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$

Note: this isn't saying that \exists a solution it's
just saying that you can't have two solutions.

Proof: assume not. Assume $u \neq v$ are in $C^2(U) \cap C(\bar{U})$
and both satisfy $\textcircled{2}$. Then $u-v$

satisfies $\begin{cases} \Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U. \end{cases}$

by maximum principle

$$\max_{\bar{U}} w = \max_{\partial U} w = 0$$

$\Rightarrow u-v \leq 0$ in U . Similarly, $v-u$ satisfies
 $\textcircled{2}$ and so $v-u \leq 0$ in U . Hence $v=u$ in U ,
as desired.

//

Finally, we prove something that won't surprise complex analysts...

Thm: if $u \in C(U)$ satisfies the mean value property

$$u(x_0) = \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} u(y) dy = \frac{1}{|\partial B(x_0, r)|} \int_{\partial B(x_0, r)} u(y) ds(y)$$

for each $x_0 \in U$ and

$B(x_0, r) \subseteq U$ then $u \in C^\infty(U)$,

Note: this theorem doesn't say anything about what $u(x)$ does as $x \rightarrow \partial U$. It doesn't ensure continuity at the boundary. This isn't surprising since

$$\Delta u = 0$$

is an internal property ... the boundary would have no effect if boundary conditions had been imposed, which they haven't.

Note: the previous note is relevant if the theorem had said "if $u \in C^2(U)$ satisfies..." since if $u \in C^2(U)$ then we know that $\Delta u = 0$. But we have something weaker here (at first sight). We have $u \in C(U) + \text{MVP} \Rightarrow u \in C^\infty(U)$

$$\text{and } u \in C^\infty(U) \Rightarrow u \in C^2(U) \Rightarrow \Delta u = 0.$$

10

proof: We will introduce a mollifier η_ε and let $u_\varepsilon = \eta_\varepsilon * u$. Now, u_ε is C^∞ if η_ε is C^∞ . So it suffices to show that $u = u_\varepsilon$ in U . Clearly we'll want to choose the mollifier wisely, to be able to play nicely w/ the mean value property. I.e. $\eta_\varepsilon = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right)$ where η is radially symmetric and $\int_{\mathbb{R}^n} \eta(x) dx = 1$. (see §C.4)

$u_\varepsilon = \eta_\varepsilon * u$ is defined in $U_\varepsilon := \{x \in U \mid \text{dist}(x, \partial U) > \varepsilon\}$

$$\begin{aligned}
 u_\varepsilon(x) &= \int_U \eta_\varepsilon(x-y) u(y) dy \\
 &= \frac{1}{\varepsilon^n} \int_U \eta\left(\frac{|x-y|}{\varepsilon}\right) u(y) dy \quad \text{since } \eta \text{ is} \\
 &\qquad \qquad \qquad \text{radially symm.} \\
 &= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \int_{\partial B(x,r)} u(y) dS(y) dr \quad \text{since } \eta \text{ has} \\
 &\qquad \qquad \qquad \text{support } \equiv B(0,1) \\
 &= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) n \alpha(n) r^{n-1} u(x) dr \quad \text{since } u \text{ satisfies} \\
 &\qquad \qquad \qquad \text{the MV property} \\
 &= u(x) \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) n \alpha(n) r^{n-1} dr \\
 &= u(x) \int_{B(0,\varepsilon)} \eta_\varepsilon(y) dy = u(x). \Rightarrow u = u_\varepsilon \text{ in } U_\varepsilon. \\
 &\qquad \qquad \qquad \Rightarrow u \in C^\infty(U_\varepsilon) \text{ for each } \varepsilon > 0 //
 \end{aligned}$$