

Welcome to Intro PDEs.

This course is intended as a basic course to help prepare you for the following Winter semester courses:

- 1) Asymptotic Methods for PDE (V. Buslaev)
- 2) Nonlinear Schrödinger Equations (J. Colliander)
- 3) General Relativity (A. Butscher)

Textbook: "Partial Differential Equations"
by L.C. Evans.

Syllabus on course web-page
<http://www.math.toronto.ca/~mpugh/>
Teaching / Mat1060/mat1060.html

Please read chapter 1 on your own. Please read §1.3 once a week for the next month.

I will introduce notation as it comes along, and will invoke facts from Real Analysis as needed. You will have to remind yourself of them or take them on faith. The text has useful appendices in this direction.

(2)

32.1 The Transport Equation

This is the simplest linear PDE one could hope for:

$$u_t + \vec{b} \cdot D u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

$\vec{b} \in \mathbb{R}^n$ fixed, $u: \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$

Notation:

1) D is the gradient of u :

$$D u = \begin{pmatrix} u_{x_1} \\ u_{x_2} \\ \vdots \\ u_{x_n} \end{pmatrix}$$

2) $u_{x_i} = \frac{\partial u}{\partial x_i}$, $u_t = \frac{\partial u}{\partial t}$

3) u is a real-valued function such that at every point in $\mathbb{R}^n \times (0, \infty)$ its derivatives satisfy the constraint

$$u_t + \vec{b} \cdot D u = 0$$

4) $x \in \mathbb{R}^n$ is considered a point in space and $t \in [0, \infty)$ is a point in time

Q: What functions on $\mathbb{R}^n \times [0, \infty)$ would satisfy the constraint

$$u_t + \vec{b} \cdot D u = 0 ?$$

A: Recall the directional derivative on \mathbb{R}^{n+1}

$$\begin{pmatrix} D u \\ u_t \end{pmatrix} \cdot \begin{pmatrix} \vec{b} \\ 1 \end{pmatrix} = \text{derivative of } u \text{ in the direction } \begin{pmatrix} \vec{b} \\ 1 \end{pmatrix}$$

So the constraint is telling us that

$u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ is such that

$$\begin{pmatrix} D_u \\ u_t \end{pmatrix} \cdot \begin{pmatrix} \vec{b} \\ 1 \end{pmatrix} = 0$$

at every point in the interior $(\mathbb{R}^n \times (0, \infty))$.

i.e. $u(x, t)$ is constant in the direction
 $\begin{pmatrix} \vec{b} \\ 1 \end{pmatrix}$.

Fix $(x, t) \in \mathbb{R}^n \times (0, \infty)$ and consider the

line $\begin{pmatrix} x + bs \\ t + s \end{pmatrix} \quad s \in \mathbb{R}$

then u should be constant along this line.

Check it:

$$\frac{d}{ds} u(x+bs, t+s) = \vec{b} \cdot Du + u_t = 0$$

because we assume
u satisfies the
PDE.

i.e. if we know the value of u at (x, t) then
we know its value anywhere on the line.

So far, we've figured out that $u_t + \vec{b} \cdot Du = 0$ is a
genuine constraint (since \exists real-valued functions on
 $\mathbb{R}^n \times (0, \infty)$ that don't satisfy it.) But \exists infinitely
many functions that do satisfy it. How can
we specify one of them?

Initial Value Problem:

$$\begin{cases} u_t + b \cdot D u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{in } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Q: Can we find u that satisfies this?

$$u(x, t) = ??$$

Idea: find the line through (x, t)
and figure out where it intersects $\mathbb{R}^n \times \{t=0\}$.

This will determine a particular value (from g) and
since u is constant on the line, this will determine
 $u(x, t)$.

$$\begin{pmatrix} x + sb \\ t + s \end{pmatrix} \cap \mathbb{R}^n \times \{t=0\}$$

$$\Rightarrow s = -t \Rightarrow x + sb = x - tb.$$

Solution $u(x, t) = g(x - tb)$ defined on $\mathbb{R}^n \times [0, \infty)$

Q1: Satisfies the PDE in $\mathbb{R}^n \times (0, \infty)$?

yes if g is C^1

Q2: Satisfies the initial data? yes!

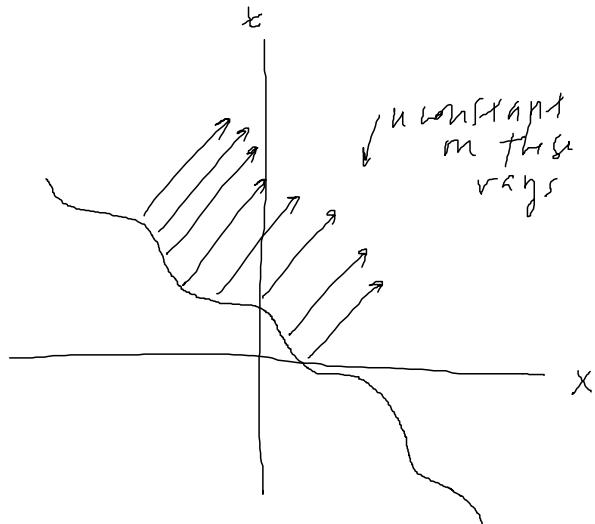
Notation:

$f \in C^1(U)$ where $U \subseteq \mathbb{R}^n$ is an open set

means: f is continuous function on U

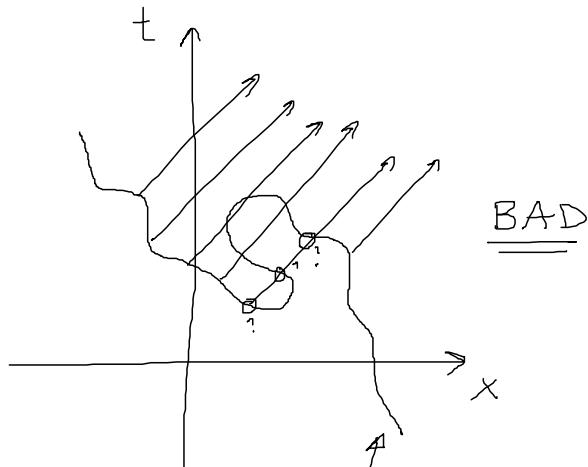
f_{x_i} is continuous function on U
for $i = 1 \dots n$

Note that we didn't really need to specify the initial data on $\mathbb{R}^n \times \{t=0\}$. We could've specified the initial data on an n -dimensional surface in \mathbb{R}^{n+1} as long as the surface was nowhere tangent to \vec{b} . E.g. for \mathbb{R}^2 $b = 1$

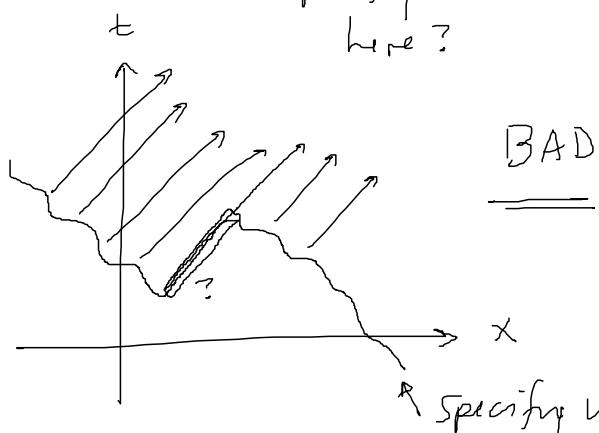


GOOD

Specify u here?



Specify u here?



Specify u here?

Nonhomogeneous problem

$$\begin{cases} u_t + \vec{b} \cdot \nabla u = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

this has the same structure as before... Try the previous approach?

Fix (x, t) and look at u on $\begin{pmatrix} x+sb \\ t+s \end{pmatrix}$

$$\frac{d}{ds} u(x+sb, t+s) = \vec{b} \cdot \nabla u + u_t = f(x+sb, t+s)$$

↑
from the PDE.

⇒ along this line, u satisfies the ODE

$$\frac{d}{ds} u(x+sb, t+s) = f(x+sb, t+s)$$

and when $s = -t$ we have the initial value $g(x-tb)$. So we've reduced the PDE to an ODE! which can be solved explicitly;

$$\begin{aligned} u(x, t) - u(x-tb, 0) &= \int_{-t}^0 \frac{d}{ds} u(x+sb, t+s) ds \\ &= \int_{-t}^0 f(x+sb, t+s) ds \\ &= \int_0^t f(x+(s-t)b, s) ds \end{aligned}$$

$$\Rightarrow u(x, t) = g(x-tb) + \int_0^t f(x+(s-t)b, s) ds. \quad \text{SOLVED!!}$$

Laplace's equation:

$$\Delta u = 0 \quad \text{on } U \quad \textcircled{4}$$

Poisson's equation:

$$-\Delta u = f \quad \text{on } U$$

Where u is a real valued function $u: \bar{U} \rightarrow \mathbb{R}$ and U is a fixed open set, f is specified, the way initial data or forcing is specified.

$$\Delta u := u_{x_1 x_1} + u_{x_2 x_2} + \dots + u_{x_n x_n}$$

is the "Laplacian" of u .

defn: a C^2 function u satisfying $\textcircled{4}$
is called a harmonic function.

notation: $u \in C^2(U)$ if

u is continuous on U

u_{x_i} is continuous on U for each $i=1\dots n$

$u_{x_i x_j}$ is continuous on U for each $i=1\dots n, j=1\dots n$

Q: Is the constraint $\Delta u = 0$ as innocuous as the constraint $u_t + b \cdot \nabla u = 0$? Tons of functions can satisfy the second PDE. How about the first?

But first, some modeling. When might such a PDE arise?

Let $V \subseteq U$ be any smooth subregion of U .

u = density of some quantity in equilibrium

F = flux density of u . If in equilibrium

then $\int_{\partial V} F \cdot \nu \, dS = 0$

ν

where ν = outward normal of ∂V . By Gauss-Green theorem,

$$0 = \int_{\partial V} F \cdot \nu \, dS = \int_V \operatorname{div}(F) \, dx$$

$\Rightarrow \operatorname{div}(F) = 0$ in U because V was arbitrary. (Note: this is true if $\operatorname{div}(F)$ is continuous in U) So all we need is some understanding of how F (the flux density of u) depends on u

assume $F = -a \nabla u$ where $a > 0$ is a fixed real #

If u = chemical concentration, then this is Fick's law of diffusion. If u = temperature, then this is Fourier's law of heat conduction. If u = electrostatic potential, then this is Ohm's law of electrical conduction.

and so $\operatorname{Div}(F) = 0 \Rightarrow F = -aDu$

yields $-\operatorname{Div}(aDu) = 0$

$\Rightarrow \Delta u = 0$ since $a \neq 0$ and

$$\operatorname{Div}(\Delta u) = \Delta u.$$

To understand Laplace's & Poisson's equations, we will seek a fundamental solution. This is an exact solution that we can use to construct other solutions from.

Fact: $\Delta u = 0$ in \mathbb{R}^n is rotation invariant.

(check this yourself!)

Idea: seek a radially symmetric solution

$$u(x) = v(r) \quad \text{where } r = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

$$\text{note } \frac{\partial r}{\partial x_i} = \frac{1}{r} (x_1^2 + \dots + x_n^2)^{-1/2} \quad 2x_i = \frac{x_i}{r}$$

$$\Rightarrow u_{x_i} = v'(r) \frac{x_i}{r}$$

$$\Rightarrow u_{x_i x_i} = v''(r) \frac{x_i^2}{r^2} + v'(r) \left(\frac{1}{r} - \frac{x_i^2}{r^3} \right)$$

$$\begin{aligned} \Rightarrow \sum_{i=1}^n u_{x_i x_i} &= v''(r) \sum \frac{x_i^2}{r^2} + v'(r) \frac{n}{r} - v'(r) \leq \frac{x_i^2}{r^2} \\ &= v''(r) + \frac{n-1}{r} v'(r) \end{aligned}$$

So if we seek $u(x) = v(r)$ such that $\Delta u = 0$, it suffices to find v that satisfies the ODE ☺

$$v'' + \frac{n-1}{r} v' = 0$$

Note: we did the PDE \Rightarrow ODE thing again! This will be a common thing when seeking special solutions.

Q: What is a "special solution"?

A: It's a solution that has an additional structure like a symmetry or the like. More later.

$$\log(|v'|)' = \frac{v''}{v'} = \frac{1-n}{r}$$

$$\Rightarrow v'(r) = \frac{a}{r^{n-1}} \quad \text{for some } a$$

$$\Rightarrow v(r) = \begin{cases} b \ln r + c & \text{if } n=2 \quad (\mathbb{R}^2) \\ \frac{b}{r^{n-2}} + c & \text{if } n \geq 3 \end{cases}$$

Q: What about $n=1$? \mathbb{R}

A: Can solve $u_{xx} = 0$ exactly... $u(x) = ax+b$

definition: the function

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln(|x|) & \text{for } n=2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & \text{for } n \geq 3 \end{cases}$$

defined for $x \neq 0$ is the fundamental solution of Laplace's equation.

Note: $\alpha(n)$ = the volume of unit ball in \mathbb{R}^n

$$= \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}$$

Note: $\Delta \Phi = 0$ holds at all $x \in \mathbb{R}^n$ except $x=0$.

Clearly Φ blows up as $x \rightarrow 0$. Is it integrable?

$n \geq 2$

$$\int_{\mathbb{R}^2} \Phi(x) dx = -\frac{2\pi}{2\pi} \left(\int_0^\infty h_n(r) r dr \right)$$

$$= \lim_{\epsilon \rightarrow 0} \left[\frac{r^2}{4} - \frac{1}{2} r^2 h_n(r) \right] \Big|_{\epsilon}^{\infty}$$

fini at bottom limit,
but not at top limit.

Lesson: $\int_K \Phi(x) dx < \infty$ for bounded sets K .

Similarly, if $n \geq 3$

$$\begin{aligned} \int_{\mathbb{R}^n} |\Phi(x)| dx &= \frac{n\alpha(n)}{n(n-2)\alpha(n)} \int_0^\infty \frac{1}{r^{n-2}} r^{n-1} dr \\ &= \frac{1}{n-2} \int_0^\infty r dr = \infty \end{aligned}$$

so Φ is integrable on bounded sets $\subset \mathbb{R}^n$ $n \geq 3$.

Also note that

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}} \quad \text{and} \quad |D^2\Phi(x)| \leq \frac{C}{|x|^n} \quad (x \neq 0)$$

for some constant $C > 0$.

Notation: $D^2u(x) = \{ D^\alpha u(x) \mid |\alpha|=2\}$

is the set of all partial derivatives of u of order 2.

Notation: $|D^2u(x)| = \sqrt{\sum_{|\alpha|=k} |D^\alpha u(x)|^2}$

Notation: if $\alpha \in \mathbb{R}^n$ is a vector where each entry is a nonnegative integer then $|\alpha| := \sum_i \alpha_i$

$$D^\alpha u(x) := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} u(x)$$

Poisson's Equation

we know that

$x \rightarrow \Phi(x)$ is harmonic for $x \neq 0$

since Δ is translation invariant,

$x \rightarrow \Phi(x-y)$ is harmonic for $x \neq y$

If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ then

$x \rightarrow f(y) \Phi(x-y)$ is harmonic for $x \neq y$

Since the sum of harmonic functions is harmonic, this suggests that

$$u(x) := \int_{\mathbb{R}^n} f(y) \Phi(x-y) dy$$

will be harmonic too. i.e.

$$\begin{aligned} \Delta u(x) &= \Delta \int_{\mathbb{R}^n} f(y) \Phi(x-y) dy \\ &= \int_{\mathbb{R}^n} f(y) \Delta \Phi(x-y) dy \\ &= 0. \end{aligned}$$

Not!!

Warning sign: since $\Delta \Phi(x-y) \sim \frac{1}{|x-y|^n}$

we see it's not integrable. In fact, we'll find that $-\Delta \Phi = \delta_0$ \Leftrightarrow the delta function.

Theorem 1: Assume $f \in C_c^2(\mathbb{R}^n)$. Let

$$u(x) := \begin{cases} -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln(|x-y|) f(y) dy & \text{if } n=2 \\ \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy & \text{if } n \geq 3 \end{cases}$$

- then (i) $u \in C^2(\mathbb{R}^n)$
(ii) $-\Delta u = f$ in \mathbb{R}^n .

Notation: $f \in C_c^2(\mathbb{R}^n)$ if $f: \mathbb{R}^n \rightarrow \mathbb{R}$ has compact support and f and all of its first and second derivatives are continuous on \mathbb{R}^n .

Proof: First prove $u \in C^2(\mathbb{R}^n)$.

u is continuous:

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy = \int_{\mathbb{R}^n} \Phi(y) f(x-y) dy$$

$$u(z) = \int_{\mathbb{R}^n} \Phi(y) f(z-y) dy$$

$$|u(x) - u(z)| = \left| \int_{\mathbb{R}^n} \Phi(y) (f(x-y) - f(z-y)) dy \right|$$

Let $K = \text{supp}\{f\}$ be compact set.

define $K_1 = \{w \in \mathbb{R}^n \mid \text{dist}(w, K) \leq 1\}$

This is another compact set.

Let $C = \int_{x-K_1} |\Phi(y)| dy$.

Given $\varepsilon > 0$ $\exists \delta > 0$ such that

$$|w-w'| < \delta \Rightarrow |f(x)-f(z)| < \frac{\varepsilon}{C}$$

for a continuous function on a compact set, so f is uniformly continuous $\Rightarrow \delta$ is independent of w and w' . Hence

$$|x-z| < \delta \Rightarrow |f(x-y) - f(z-y)| < \frac{\varepsilon}{C} \text{ for all } y.$$

Let $\tilde{\delta} = \min\{\delta, 1\}$. Then

$$|x-z| < \tilde{\delta} \Rightarrow |f(x-y) - f(z-y)| < \frac{\varepsilon}{C}$$

and $|x-z| < 1$

$$|u(x) - u(z)| = \left| \int_{x-K_1} \Phi(y) (f(x-y) - f(z-y)) dy \right|$$

Valid since $|x-z| < 1 \Rightarrow f(z-y) = 0$ for $y \notin x-K_1$, and $f(x-y) = 0$ for $y \notin x-K_1$ hence for $y \notin x-K_1$,

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$$\begin{aligned}
 \Rightarrow |u(x) - u(z)| &\leq \int_{x-K_1} |\Phi(y)| |f(x-y) - f(z-y)| dy \\
 &\leq \int_{x-K_1} |\Phi(y)| \frac{\varepsilon}{c} dy \quad \text{since } |x-z| < \delta \\
 &= \frac{\varepsilon}{c} \int_{x-K_1} |\Phi(y)| dy = \frac{\varepsilon}{c} \cdot c = \varepsilon.
 \end{aligned}$$

This proves that u is continuous at x_0 as desired.

Now to prove that u_{x_i} exists. We need to show that for each $x \in \mathbb{R}^n$, the following limit exists: $\lim_{h \rightarrow 0} \frac{u(x+he_i) - u(x)}{h}$,

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{u(x+he_i) - u(x)}{h} &= \lim_{h \rightarrow 0} \int_{\mathbb{R}^n} \Phi(y) \frac{f(x+he_i-y) - f(x-y)}{h} dy \\
 &= \lim_{h \rightarrow 0} \int_{x-K_1} \Phi(y) \frac{f(x+he_i-y) - f(x-y)}{h} dy
 \end{aligned}$$

To take the limit under the integral sign, I will use the Lebesgue Dominated Convergence Theorem

LDC theorem: Assume the functions $\{\psi_n\}_1^\infty$ are integrable and

$\psi_n(x) \rightarrow \psi(x)$ for almost all x .

Assume $|\psi_n(x)| \leq g(x)$ for almost all x

where $\int g(y) dy < \infty$. Then

$$\lim_{n \rightarrow \infty} \int \psi_n(y) dy = \int \psi(y) dy.$$

]

To apply the LDC, let $\{h_n\}_1^\infty$ be a sequence in \mathbb{R} such that $h_n \rightarrow 0$. Let

$$\psi_n(x) = \Phi(y) \frac{f(x+h_n e_i - y) - f(x-y)}{h_n}$$

then $\psi_n(x) \rightarrow \Phi(y) \frac{\partial f}{\partial x_i}(x-y)$ at all y except $y=0$

Now, f_{x_i} is a continuous function on a compact set so $M = \max_{\mathbb{R}} |f_{x_i}(y)|$ exists.

$$\text{Let } g(y) = M |\Phi(y)|$$

By the mean value theorem,

$$\psi_n(y) \leq |\Phi(y)| M = g(y) \text{ at all } y$$

$$\text{and } \int_{V,K} g(y) dy = M \int_{X-K_1} |\Phi(y)| dy < \infty.$$

So the Lebesgue Dominated Convergence theorem applies and

$$\lim_{k \rightarrow \infty} \int_{x-K_1} \Phi(y) \frac{f(x+h_k e_i - y) - f(x-y)}{h_k} dy = \int_{x-K_1} \Phi(y) \frac{\partial f}{\partial x_i}(x-y) dy$$

Now argue that this implies

$$\lim_{h \rightarrow 0} \int_{x-K_1} \Phi(y) \frac{f(x+he_i - y) - f(x-y)}{h} dy = \int_{x-K_1} \Phi(y) \frac{\partial f}{\partial x_i}(x-y) dy$$

This proves that $\frac{\partial u}{\partial x_i}(x)$ exists at all $x \in \mathbb{R}^n$.

Now prove that $\frac{\partial u}{\partial x_i}$ is continuous on \mathbb{R}^n .

Now prove that $\frac{\partial^2 u}{\partial x_i \partial x_j}(x)$ exists at all $x \in \mathbb{R}^n$

Now prove that $\frac{\partial^2 u}{\partial x_i \partial x_j}$ is continuous on \mathbb{R}^n .

Done with part (i)

Now to show

that $-\Delta u = f$ in \mathbb{R}^n . We want to somehow use the fact that $\Delta \underline{\Phi} = 0$ everywhere except at $|x|=0$.

We will also want to take advantage of its radial symmetry. And will have to deal with its singularity at $|x|=0$. Do this via a limiting process.

Fix $\Sigma > 0$.

$$\Delta u(x) = \int_{B(0, \Sigma)} \underline{\Phi}(y) \Delta_x f(x-y) dy + \int_{\mathbb{R}^n - B(0, \Sigma)} \underline{\Phi}(y) \Delta_x f(x-y) dy$$

$\underbrace{\hspace{10em}}$ $\underbrace{\hspace{10em}} \rightarrow$

I_ε J_ε

$$\begin{aligned}
 |I_\varepsilon| &= \left| \int_{B(0, \varepsilon)} \Phi(y) \Delta_x f(x-y) dy \right| \\
 &\leq \int_{B(0, \varepsilon)} |\Phi(y)| |\Delta_x f(x-y)| dy \quad \text{if } \varepsilon < 1 \quad (\text{since for } n=2 \\
 &\quad \Phi \text{ is negative if } |y| > 1.) \\
 &\leq C \|D^2 f\|_{L^\infty(\mathbb{R}^n)} \int_{B(0, \varepsilon)} \Phi(y) dy \quad \text{know } f \in C^2_c(\mathbb{R}^n) \\
 &\quad \Rightarrow \Delta_x f(x-y) \text{ is uniformly bounded} \\
 &\leq \begin{cases} C \varepsilon^2 |\ln \varepsilon| & \text{if } n=2 \\ C \varepsilon^n & \text{if } n \geq 3. \end{cases}
 \end{aligned}$$

We see that as $\varepsilon \downarrow 0$ $I_\varepsilon \rightarrow 0$ ☺

$$\begin{aligned}
 I_\varepsilon &= \int_{\mathbb{R}^n - B(0, \varepsilon)} \Phi(y) \Delta_x f(x-y) dy = \int_{\mathbb{R}^n - B(0, \varepsilon)} \Phi(y) \Delta_y f(x-y) dy \\
 &= - \int_{\mathbb{R}^n - B(0, \varepsilon)} D\Phi(y) \cdot D_y f(x-y) dy + \int_{\partial B(0, \varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y) dy
 \end{aligned}$$

where ν is the unit normal vector that points towards the origin

$$= K_\varepsilon + L_\varepsilon$$

Note: used integration by parts / the Gauss-Green thm. this is valid since Φ is smooth in $\mathbb{R}^n - B(0, \varepsilon)$.

$$\begin{aligned}
 |L_\varepsilon| &= \left| \int_{\partial B(0, \varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y) dS(y) \right| \\
 &\leq \|Df\|_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0, \varepsilon)} \Phi(y) dS(y) \\
 &= \|Df\|_{L^\infty(\mathbb{R}^n)} \Phi(\varepsilon) \int_{\partial B(0, \varepsilon)} dS(y) \\
 &= \|Df\|_{L^\infty(\mathbb{R}^n)} \Phi(\varepsilon) n \alpha(n) \varepsilon^{n-1} \leq \begin{cases} C \varepsilon \ln \varepsilon & n=2 \\ C \varepsilon & n \geq 3 \end{cases}
 \end{aligned}$$

So $L_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

On to handling K_ε

$$\begin{aligned}
 K_\varepsilon &= - \int_{\mathbb{R}^n - B(0, \varepsilon)} D\Phi(y) \cdot D_y f(x-y) dy \\
 &= \int_{\mathbb{R}^n - B(0, \varepsilon)} \Delta \Phi(y) f(x-y) dy - \int_{\partial B(0, \varepsilon)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y) dy \\
 &= - \int_{\partial B(0, \varepsilon)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y) dy \quad \text{since } \Delta \Phi = 0 \text{ in } \mathbb{R}^n - B(0, \varepsilon)
 \end{aligned}$$

Know $D\Phi(y) = \frac{-1}{n \alpha(n)} \frac{y}{|y|^n}$ and $\nu = \frac{-y}{|y|}$

$$\Rightarrow \frac{\partial \phi}{\partial \nu} = \left(\frac{-1}{n\alpha(n)} \frac{y}{|y|^n} \right) \cdot \left(-\frac{y}{|y|} \right) = \frac{1}{n\alpha(n)} \frac{1}{|y|^{n-1}}$$

$$\Rightarrow K_\varepsilon = -\frac{1}{n\alpha(n)} \frac{1}{\varepsilon^{n-1}} \int_{\partial B(0, \varepsilon)} f(x-y) dS(y)$$

$$= -\frac{1}{\text{surface area of } \partial B(0, \varepsilon)} \int_{\partial B(0, \varepsilon)} f(x-y) dS(y)$$

$$= -f \int_{\partial B(0, \varepsilon)} f(x-y) dS(y) \quad f := \text{mean value.}$$

Okay. Now we have

$$\Delta u(x) = I_\varepsilon + L_\varepsilon - \int_{\partial B(0, \varepsilon)} f(x-y) dS(y)$$

Take $\varepsilon \rightarrow 0$. I_ε and $L_\varepsilon \rightarrow 0$ and $\int_{\partial B(0, \varepsilon)} f(x-y) dS(y)$

goes to $f(x)$ Hence

$$\Delta u(x) = -f(x), \text{ as desired. //}$$