

Oct 8, 2004

From last time, we have a strong maximum principle for the heat equation. From this, it follows:

Corr: if $U \subseteq \mathbb{R}^n$ is open, bounded, and connected and $u \in C^2(U_T) \cap C(\bar{U}_T)$ satisfies

$$\begin{cases} u_t = \Delta u & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0, T] \\ u = g & \text{on } U \times \{t=0\} \end{cases}$$

and $g \geq 0$. If $\exists \tilde{x}_0 \in U$ such that $g(\tilde{x}_0) > 0$ then $u > 0$ on U_T .

proof: Note that the strong max principle also yields a strong minimum principle.

Assume since $u = 0$ on $\partial U \times [0, T]$ and $g \geq 0$ on $U \times \{t=0\}$,

$$\min_{\bar{U}_T} u = \min_{\Gamma_T} u = 0$$

If $\exists (x_0, t_0) \in U_T$ such that $u(x_0, t_0) = 0$ then $u \equiv 0$ on \bar{U}_{t_0} and hence $g \equiv 0$. This contradicts $g(\tilde{x}_0) > 0$. //

Just as the maximum principle for Laplace eqn gave uniqueness for Poisson's eqn, the maximum principle for the heat eqn gives uniqueness for the nonhomogeneous heat eqn;

thm: Let $U \subseteq \mathbb{R}^n$ be open + bounded. Let $g \in C(\Gamma_T)$ and $f \in C(U_T)$. Then \exists at most one solution $u \in C^2_1(U_T) \cap C(\bar{U}_T)$ of

$$\begin{cases} u_t = \Delta u + f & \text{in } U_T \\ u = g & \text{on } \Gamma_T \end{cases}$$

the initial/boundary value problem:

proof: Assume \exists two solutions u_1, u_2 in $C^2_1(U_T) \cap C(\bar{U}_T)$. Apply the maximum principle to $u_1 - u_2$ and to $u_2 - u_1$. //

Now we prove a maximum principle for the Cauchy problem. i.e. $u_t = \Delta u$ on $\mathbb{R}^n \times (0, T)$ with specified initial data

We'll have to make some assumptions on the behavior at infinity

Thm: (maximum principle for the Cauchy problem)

Suppose $u \in C^2_1(\mathbb{R}^n \times (0, T)) \cap C(\mathbb{R}^n \times [0, T])$

solves
$$\begin{cases} u_t = \Delta u & \text{in } \mathbb{R}^n \times (0, T) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

and satisfies the growth estimate

$$u(x, t) \leq A e^{a|x|^2} \quad x \in \mathbb{R}^n, 0 \leq t \leq T$$

for constants $A, a > 0$. Then

$$\sup_{\mathbb{R}^n \times [0, T]} u = \sup_{\mathbb{R}^n} g$$

Note: even though it's described as a "maximum" principle, it's really a supremum principle. The suprema above could be infinite.

Scary Fact: There are infinitely many solutions

of
$$\begin{cases} u_t = \Delta u & \text{in } \mathbb{R}^n \times (0, T) \\ u = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Each solution (other than the $u=0$ solution) grows very rapidly as $|x| \rightarrow \infty$. The growth bound $Ae^{a|x|^2}$ allows us to exclude "nonphysical" solutions.

(4)

Proof: We will prove the result by first assuming T is "small" and will then generalize to unrestricted $T < \infty$.

Assume $T < \frac{1}{4a}$. Choose $\varepsilon > 0$ such that

$T + \varepsilon < \frac{1}{4a}$. Fix $y \in \mathbb{R}^n$ and $\mu > 0$ and

let

$$v(x, t) := u(x, t) - \frac{\mu}{(T + \varepsilon - t)^{\mu/2}} e^{-\frac{|x-y|^2}{4(T+\varepsilon-t)}}$$

note 1: since u is a solution of $u_t = \Delta u$ on $\mathbb{R}^n \times (0, T)$
 v is also a solution of $v_t = \Delta v$ on $\mathbb{R}^n \times (0, T)$

note 2: v is "close" to u at $x=y$:

$v(y, t) = u(y, t) - \frac{\mu}{(T + \varepsilon - t)^{\mu/2}}$. The smaller μ is, the closer $v(y, t)$ is to $u(y, t)$.

note 3: $4(T + \varepsilon - t) < 4(T + \varepsilon) < \frac{1}{a}$ and so

$e^{a|x|^2}$ grows more slowly as $|x| \rightarrow \infty$ than

$e^{-\frac{|x-y|^2}{4(T+\varepsilon-t)}}$ does (Fix y and t) so since

$u(x, t) \leq A e^{a|x|^2}$ we have introduced fast decay at ∞ so that even though u might grow as $|x| \rightarrow \infty$, v most assuredly does not.

Fix $r > 0$ and let $U = B(y, r)$ and $U_T = B(y, r) \times (0, T]$

Since u solves the heat equation in U_T and is in $C^2_1(U_T) \cap C(\bar{U}_T)$, the maximum principle applies:

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u$$

we now analyze u on $\Gamma_T = \{\partial B(y, r) \times [0, T]\} \cup \overline{B(y, r)} \times \{t=0\}$

on $\overline{B(y, r)} \times \{t=0\}$:

$$\begin{aligned} u(x, 0) &= u(x, 0) - \frac{\mu}{(T+\epsilon)^{n/2}} e^{\frac{|x-y|^2}{4(T+\epsilon)}} \\ &= f(x) - \frac{\mu}{(T+\epsilon)^{n/2}} e^{\frac{|x-y|^2}{4(T+\epsilon)}} < f(x) \end{aligned}$$

$$\Rightarrow \max_{\overline{B(y, r)} \times \{t=0\}} u < \sup_{\mathbb{R}^n} f$$

on $\partial B(y, r) \times [0, T]$:

$$\begin{aligned} u(x, t) &= u(x, t) - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\epsilon-t)}} \\ &\leq A e^{a|x|^2} - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\epsilon-t)}} \\ &\leq A e^{a(|y|+r)^2} - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\epsilon-t)}} \end{aligned}$$

$\leq \sup_{\mathbb{R}^n} f$ if r chosen large enough. (by note 3)

this shows that for v chosen sufficiently large,

$$\max_{\Gamma_T} u \leq \sup_{\mathbb{R}^n} g.$$

And so
$$\frac{\max}{V_T} u \leq \sup_{\mathbb{R}^n} g.$$

Specifically, since $(y, t) \in \overline{U_T}$,

$$u(y, t) \leq \sup_{\mathbb{R}^n} g$$

And so

~~***~~
$$u(y, t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} < \sup_{\mathbb{R}^n} g \quad (\text{by note 2})$$

Recall that y and μ are fixed but arbitrary.

Since ~~***~~ holds $\forall \mu > 0$, we can take $\mu \rightarrow 0$

and find
$$u(y, t) \leq \sup_{\mathbb{R}^n} g.$$
 Since y

was arbitrary, we then find

$$\sup_{\mathbb{R}^n \times [0, T]} u \leq \sup_{\mathbb{R}^n} g \quad \text{as desired.}$$

Finally, if T is "small", apply the above argument on a sequence of time intervals

$$[0, T_1], [T_1, T_2], [T_2, T_3], \dots, [T_{n-1}, T_n]$$

where $T_n \leq T$ and $T_n - T_{n-1} < \frac{1}{4a}$.

Corr: (Uniqueness for the Cauchy Problem)

Let $g \in C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n \times [0, T])$. Then there exists at most one solution $u \in C_1^2(\mathbb{R}^n \times (0, T]) \cap C(\mathbb{R}^n \times [0, T])$ of the initial value problem

$$\begin{cases} u_t = \Delta u + f & \text{in } \mathbb{R}^n \times (0, T) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

satisfying the

$$|u(x, t)| \leq A e^{a|x|^2} \quad \text{on } \mathbb{R}^n \times [0, T]$$

for some $A, a > 0$.

Proof:

Assume there are two solutions u_1, u_2 satisfying growth estimate

Apply previous theorem to $w = u_1 - u_2$ and $w = u_2 - u_1$, conclude $w \equiv 0$ //

Recall that

$$u \in C^2(U) \text{ and } \Delta u = 0 \Rightarrow u \in C^\infty(U) \\ \Rightarrow u \text{ analytic in } U.$$

here we have

Theorem: Suppose $u \in C^2_1(U_T)$ solves the heat equation in U_T . Then $u \in C^\infty(U_T)$.

proof: see book (Thm 8).

thm: Suppose $u \in C^2_1(U_T)$ solves the heat equation in U_T . Fix $t \in (0, T]$
Then $x \rightarrow u(x, t)$ is analytic in U .

proof: see Mikhailov.

note: If you fix $x \in U$ and ask if
 $t \rightarrow u(x, t)$ is analytic in $(0, T]$,
it's not necessarily true.

We'll now apply some of my favorite methods:
Energy methods.

Energy Methods.

Assume \mathcal{U} is open + bounded and $\partial\mathcal{U}$ is C^1 . This will allow us to apply the divergence theorem to \mathcal{U} .

$$\text{Consider } \begin{cases} u_t = \Delta u + f & \text{in } \mathcal{U}_T \\ u = g & \text{on } \Gamma_T \end{cases}$$

where $T > 0$ is fixed.

thm: If \mathcal{U} is open and bounded and $\partial\mathcal{U}$ is C^1 then $\textcircled{*}$ has at most one solution in $C_1^2(\overline{\mathcal{U}_T})$

proof: Assume you have two solutions u and \tilde{u} . Let $w = u - \tilde{u}$. Then w solves

$$\begin{cases} w_t = \Delta w & \text{in } \mathcal{U}_T \\ w = 0 & \text{on } \Gamma_T \end{cases}$$

Let $e(t) = \int_{\mathcal{U}} w(x,t)^2 dx$ be the "energy" of w at time t . Then

$$\begin{aligned} \frac{d}{dt} e(t) &= \frac{d}{dt} \int_{\mathcal{U}} w(x,t)^2 dx = 2 \int_{\mathcal{U}} w(x,t) \frac{\partial w}{\partial t}(x,t) dx \\ &= 2 \int_{\mathcal{U}} w(x,t) \nabla \cdot \nabla w(x,t) dx \end{aligned}$$

$$\Rightarrow \frac{de}{dt} = -2 \int_U \nabla w \cdot \nabla w \, dx + 2 \int_{\partial U} w(x,t) \nabla w(x,t) \cdot \nu(x,t) \, dS(x)$$

$$= -2 \int_U |\nabla w(x,t)|^2 \, dx \quad \text{since } w(x,t) \equiv 0 \text{ on } \partial U.$$

$$\Rightarrow \frac{de}{dt} \leq 0 \quad \Rightarrow \quad e(t) \leq e(0) = \int w(x,0)^2 \, dx = 0$$

∴ $0 \leq e(t) \leq e(0) = 0$ for all $t \in [0, T]$

This shows that $e(t) = 0$ at each time.

Since $x \rightarrow u(x,t)$ is in $C(\bar{U})$

this shows $w(x,t) \equiv 0$ on $C(\bar{U})$

as desired. //

So we have forwards uniqueness. What about backwards uniqueness?

Consider two solutions u and \tilde{u} which have the same boundary conditions.

$$u \text{ solves } \begin{cases} u_t = \Delta u \text{ in } U_T \\ u = g \text{ on } \partial U \times [0, T] \end{cases} \quad \tilde{u} \text{ solves } \begin{cases} \tilde{u}_t = \Delta \tilde{u} \text{ in } U_T \\ \tilde{u} = g \text{ on } \partial U \times [0, T] \end{cases}$$

theorem: Assume Ω is open + bounded (11)
 and $\partial\Omega$ is C^1 . Let u and \tilde{u} be two solutions
 of $u_t = \Delta u$ in Ω_T and $u, \tilde{u} \in C^2(\overline{\Omega_T})$
 assume they agree on $\partial\Omega \times [0, T]$:
 $u = \tilde{u} = g$ on $\partial\Omega \times [0, T]$.

If $u(x, T) \equiv \tilde{u}(x, T)$ for all $x \in \Omega$
 then $u \equiv \tilde{u}$ in Ω_T .

proof: Let $w = u - \tilde{u}$. Then w is a
 solution of

$$\begin{cases} w_t = \Delta w & \text{in } \Omega_T \\ w = 0 & \text{on } \partial\Omega \times [0, T] \\ w = 0 & \text{on } \Omega \times \{t = T\} \end{cases}$$

let $e(t) = \int_{\Omega} w(x, t)^2 dx$.

As before,

$$\frac{de}{dt} = -2 \int_{\Omega} |\nabla w(x, t)|^2 dx$$

$$\Rightarrow \frac{d^2 e}{dt^2} = -4 \int_{\Omega} \nabla w(x, t) \cdot \nabla w_t(x, t) dx$$

$$= 4 \int_{\Omega} \operatorname{Div}(\nabla w) w_t(x, t) dx$$

$$- 4 \int_{\partial\Omega} \nabla w(x, t) \cdot \nu w_t(x, t) dx$$

note: $w_t \equiv 0$ on ∂U .

$$\Rightarrow \frac{d^2 e}{dt^2} = 4 \int_U (\Delta w(x, t))^2 dx$$

on the other hand, $\int_U |Dw|^2 dx = - \int_U w \Delta w dx$

$$\leq \sqrt{\int_U w^2 dx} \sqrt{\int_U (\Delta w)^2 dx}$$

$$\begin{aligned} \Rightarrow \left(\frac{de}{dt}\right)^2 &= 4 \left(\int_U |Dw|^2 dx\right)^2 \\ &\leq 4 \int_U w^2 dx \int_U (\Delta w)^2 dx = e(t) \frac{d^2 e}{dt^2} \end{aligned}$$

And so we've proven a differential inequality:

$$\left(\frac{de}{dt}\right)^2 \leq e(t) \frac{d^2 e}{dt^2} \quad \text{on } [0, T]. \quad \textcircled{*}$$

our goal is to prove that $e(t) \equiv 0$ on $[0, T]$

we know $e(T) = 0$. Assume $e \not\equiv 0$ on $[0, T]$

then $\exists t_0$ s.t. that $e(t_0) > 0$. $\Rightarrow \exists$ an

interval $[t_1, t_2] \subseteq [0, T]$ s.t. that $t_0 \in [t_1, t_2]$

and $e > 0$ on $[t_1, t_2)$ but $e(t_2) = 0$.

(This is true since $e(t)$ is continuous on $[0, T]$
since $w \in C^2(\bar{U}_T)$.)

Let $f(t) := \ln(e(t))$ defined on $[t_1, t_2]$

$$\text{Then } \frac{df}{dt} = \frac{1}{e(t)} \frac{de}{dt} \Rightarrow \frac{d^2f}{dt^2} = -\frac{1}{e(t)^2} \left(\frac{de}{dt}\right)^2 + \frac{1}{e(t)} \frac{d^2e}{dt^2}$$

$$\begin{aligned} \Rightarrow \frac{d^2f}{dt^2} &= \frac{e(t)}{e(t)^2} \frac{d^2e}{dt^2} - \frac{1}{e(t)^2} \left(\frac{de}{dt}\right)^2 \\ &= \frac{1}{e(t)^2} \left[e(t) \frac{d^2e}{dt^2} - \left(\frac{de}{dt}\right)^2 \right] \geq 0 \quad \text{by } \textcircled{7} \end{aligned}$$

This shows that

$f(t)$ is convex on $[t_1, t_2]$

\Rightarrow if $t \in (t_1, t_2)$ then for any $\tau \in (0, 1)$

$$f(\tau t + (1-\tau)t_1) \leq \tau f(t) + (1-\tau)f(t_1)$$

$$\Rightarrow \ln(e(\tau t + (1-\tau)t_1)) \leq \tau \ln(e(t)) + (1-\tau) \ln(e(t_1))$$

$$\Rightarrow e(\tau t + (1-\tau)t_1) \leq e(t)^\tau e(t_1)^{1-\tau}$$

$$\Rightarrow 0 \leq e(\tau t_2 + (1-\tau)t_1) \leq e(t_2)^\tau e(t_1)^{1-\tau}$$

Note: I fixed τ and took t to t_2 . I couldn't do this at the level of $f(t)$ since $f(t) \rightarrow -\infty$ as $t \rightarrow t_2$. I used f to prove an inequality for e on $[t_1, t_2)$ and then used the continuity of $e(t)$ to extend the inequality to $[t_1, t_2]$

Since $e(t_2) = 0$ by assumption,

$$e(\tau t_2 + (1-\tau)t_1) = 0 \quad \text{for each } \tau \in (0, 1)$$

$\Rightarrow e \equiv 0$ on $[t_1, t_2]$ contradicting $e > 0$ on $[t_1, t_2)$ //

Should we be surprised by this result?
That a solution is fully determined by its boundary data and its distribution at the final time?

You're not shocked if you've ever solved the heat equation by separation of variables.

For example, if you're solving $u_t = u_{xx}$ on $[0, 2\pi]$ with initial data $g(x)$ then

$$g(x) = \sum_{-\infty}^{\infty} g_k e^{ikx} \quad \text{and}$$

$$u(x,t) = \sum_{-\infty}^{\infty} g_k e^{-k^2 t} e^{ikx}$$

is the solution. On the other hand, if you'd been given the solution at time $T: u(x,T) = h(x)$

then if $h(x) = \sum_{-\infty}^{\infty} h_k e^{ikx}$ the solution at time t would be

$$u(x,t) = \sum_{-\infty}^{\infty} h_k e^{k^2(T-t)} e^{ikx}$$

and so you can fully determine the solution from the solution at time $t = T$.

So you're not shocked to learn that if you know $u(x, T)$ then you can reconstruct $u(x, t)$ at earlier times.

However, you should note something from the solution:

$$u(x, t) = \sum_{-\infty}^{\infty} h_k e^{k^2(T-t)} e^{ikx}$$

if you have a small mis-measurement in the data at time T , then this error will grow exponentially!

Another way to say this: $u(x, t)$ is not a continuous function of the data h at time T . If it were then given $\varepsilon > 0$ $\exists \delta > 0$ such that

$$\|u(x, T) - \tilde{u}(x, T)\|_{L^\infty} < \delta \Rightarrow \|u(x, t) - u(x, \tilde{t})\|_{L^\infty} < \varepsilon.$$

No matter how small you take δ , you can find a $\tilde{u}(x, T)$, which is within δ of $u(x, T)$, such that $\|\tilde{u}(x, t) - u(x, t)\|_{L^\infty} > \varepsilon$. This is what mathematicians mean when they say the backwards heat equation is "ill-posed". Physicists/engineers would say that the backwards heat equation catastrophically magnifies noise/error.

Note: For general open, bounded domains with smooth boundaries there is an analogue of fourier expansions. And solutions can often be written as

$$u(x,t) = \sum_0^\infty e^{-\lambda_n t} \phi_n(x)$$

where ϕ_n are eigenfunctions and λ_n are eigenvalues of the Laplacian. $\lambda_n \rightarrow \infty$ like k^2 often. And so the prior observations on reversibility + ill-posedness hold.

Energy If $u(x,t) = \sum_{-\infty}^\infty u_n(t) e^{ikx}$ then

$$E(t) := \int (u(x,t))^2 dx = \sum_{-\infty}^\infty |u_n(t)|^2$$

and so since $u_n(t) = e^{-k^2 t}$,

$$E(t) = \sum_{-\infty}^\infty (e^{-k^2 t} g_n)^2$$

In the backwards-time proof, we showed that when the energy is positive, it's log-convex:

$$E(\tau t_1 + (1-\tau)t_2) \leq E(t_1)^\tau E(t_2)^{1-\tau}$$

Note that if you look at the energy held in a single mode : e^{ikx}

then the log-convexity is equality :

$$\begin{aligned} \text{energy in } k^{\text{th}} \text{ mode at time } \tau t_1 + (1-\tau)t_2 \\ = \left(e^{-k^2 [\tau t_1 + (1-\tau)t_2]} g_k \right)^2 \end{aligned}$$

$$\text{energy in } k^{\text{th}} \text{ mode at time } t_1 : \left(e^{-k^2 t_1} g_k \right)^2$$

$$\text{at time } t_2 : \left(e^{-k^2 t_2} g_k \right)^2$$

$$\left(e^{-k^2 [\tau t_1 + (1-\tau)t_2]} g_k \right)^2 \stackrel{?}{\leq} \left(e^{-k^2 t_1} g_k \right)^{2\tau} \left(e^{-k^2 t_2} g_k \right)^{2(1-\tau)}$$

$$e^{-2k^2 [\tau t_1 + (1-\tau)t_2]} \stackrel{?}{\leq} e^{-2\tau k^2 t_1} e^{-2(1-\tau)k^2 t_2}$$

holds identically at single mode level. So if your initial data is a single mode then the log-convexity inequality is equality

You should think about how the sums affect things:

$$\left(\sum_{-\infty}^{\infty} e^{-2k^2 [\tau t_1 + (1-\tau)t_2]} g_k^2 \right) \leq \left(\sum_{-\infty}^{\infty} e^{-2k^2 t_1} g_k^2 \right)^{\tau} \left(\sum_{-\infty}^{\infty} e^{-2k^2 t_2} g_k^2 \right)^{1-\tau}$$