

Oct 6, 2004

(1)

Thm: Assume  $U$  is open and bounded in  $\mathbb{R}^n$ .

Assume  $u \in C_1^2(U_T) \cap C(\bar{U}_T)$  solves the heat equation in  $U_T$ .

i) Then  $\max_{\bar{U}_T} u = \max_{\Gamma_T} u$

ii) If  $U$  is connected and  $\exists (x_0, t_0) \in U_T$  such that  $u(x_0, t_0) = \max_{\bar{U}_T} u$  then

$u$  is constant in  $\overline{U_{t_0}}$ .

Note 1: Recall that  $\Gamma_T = U_x \cup \partial U \times [0, T] \setminus \{t=0\}$

i.e. the bottom & sides of the parabolic cylinder but not the top. So the "hottest point" must occur in the initial data or on the boundary of the region  $U$  at some time  $t \in [0, T]$

Note 2: Because of causality, having the maximum at  $(x_0, t_0)$  cannot influence  $u(x, t)$  for  $t > t_0$ .

If  $u$  has a maximum at  $(x_0, t_0)$  where  $x_0 \in U$

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and  $U$  is connected, then we wonder that  $u = \text{constant}$  on  $\overline{U_{t_0}}$ . The solution  $u$  at  $(x, t)$  where  $t > t_0$  will solve

$$\begin{cases} u_t = \Delta u & \text{in } U \times (t_0, T) \\ u = \text{const} & \text{in } U \times \{t = t_0\} \\ u = g & \text{in } \partial U \times (t_0, T) \end{cases}$$

where the boundary condition  $g$  can make  $u$  non-constant even though  $u$  was constant in  $\overline{U_{t_0}}$ .

Proof:

Assume  $\exists (x_0, t_0) \in U_T$  such that

$$u(x_0, t_0) = M := \max_{\overline{U_T}} u$$

Then for sufficiently small  $r > 0$ ,  $E(x_0, t_0; r) \subseteq U_T$  and by the MVP for the heat equation

$$M = u(x_0, t_0) = \frac{1}{4r^n} \iint_{E(x_0, t_0; r)} u(y, s) \frac{|x-y|^2}{(t-s)^2} dy ds \leq M;$$

Since  $\frac{1}{4r^n} \iint_{E(x_0, t_0; r)} \frac{|x-y|^2}{(t-s)^2} dy ds = 1$ . Since  $u$  is continuous,

equality holds iff  $u$  is equal to  $M$  in  $E(x_0, t_0; r)$

Thus  $u \equiv M$  in  $E(x_0, t_0; r)$ .

Now let  $(y_0, s_0)$  be another point in  $U_T$  with  $s_0 < t_0$ . Connect  $(y_0, s_0)$  to  $(x_0, t_0)$  by a line segment,  $L$ . Let

$$r_0 = \inf \left\{ s \geq s_0 \mid u(x, t) = M \text{ for all } (x, t) \in L \text{ with } s \leq t \leq t_0 \right\}$$

$u$  is continuous hence  $u(z_0, r_0) = M$  for some point  $(z_0, r_0) \in L \cap U_T$ . We want to show that  $r_0 = s_0$ .

and hence  $u(y_0, s_0) = M$ . Assume not. Assume  $r_0 > s_0$ .  $u \equiv M$  on  $E(z_0, r_0; \tilde{r})$

for some sufficiently small  $\tilde{r}$ . Hence  $r_0$  cannot be the infimum since

$$r_0 > s_0 \text{ and } u \equiv M \text{ on } E(z_0, r_0; \tilde{r})$$

implies one can find a  $(z_1, r_1) \in L \cap U_T$  with  $r_1 < r_0$  and  $u(z_1, r_1) = M$ .  $\cancel{\text{Hence }} r_0 = s_0$

and  $u(y_0, s_0) = M$ .

This proves that if  $u(x_0, t_0) = M$  then  $u(y_0, s_0) = M$  for any  $(y_0, s_0)$  with  $s_0 < t_0$ . If

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$(y_0, s_0)$  can be connected  
to  $(x_0, t_0)$  by a line segment

Next step. If  $U$  is connected then for any  $(x, t) \in U_+$  with  $t < t_0 \exists$  a finite set of line segments connecting  $(x, t)$  to  $(x_0, t_0)$ . Why? The set of  $x \in U$  that can be connected to  $x_0$  by a polygonal path is nonempty, open, and relatively closed in  $U$ . So it's all of  $U$ . Now, lift the polygonal path in  $U$  to a polygonal path in  $U_+$  connecting  $(x, t)$  to  $(x_0, t_0)$  by introducing intermediate times  $t_i$ :

$$t = t_m < t_{m-1} < \dots < t_1 < t_0$$

where  $m = \#$  of segments in polygonal path.

By the previous part,  $u(x, t) = M$  since  $u(x_i, t_i) = M$  for  $i = 1 \dots m$ .

Now this proves that  $u \equiv M$  on  $U_{t_0}$ . To get that  $u \equiv M$  on  $\overline{U}_{t_0}$  we use the fact that  $u \in C(\overline{U}_T)$  and  $\Sigma u$  is constant on the closure of  $U_{t_0}$ .

This proves the second part of the maximum principle. Now to prove the first part. It's just like the max. principle for Laplace's eqn.

Assume  $\max_{\overline{U}_T} u > \max_{\Gamma_T} u$ . Then  $\exists (x_0, t_0) \in U_T$

such that  $u(x_0, t_0) = M = \max_{\overline{U}_T} u$ . Let  $V$  be the connected component of  $U$  containing  $x_0$ . By what we've already done,  $u \equiv M$  on  $\overline{V}_{t_0}$ .

Since  $\partial \overline{V}_{t_0} - V_{t_0} - \{V \times \{t=t_0\}\} \subseteq \Gamma_T$  we have  $\max_{\partial V_{t_0}} u \leq \max_{\Gamma_T} u$ . But  $\max_{\partial V_{t_0}} u = M$

contradicting  $\max_{\Gamma_T} u < M$ . Thus  $\max_{\overline{U}_T} u = \max_{\Gamma_T} u$