

Characteristics

Recall the first day of class, we solved

$$u_t + \vec{b} \cdot \nabla u = 0$$

and

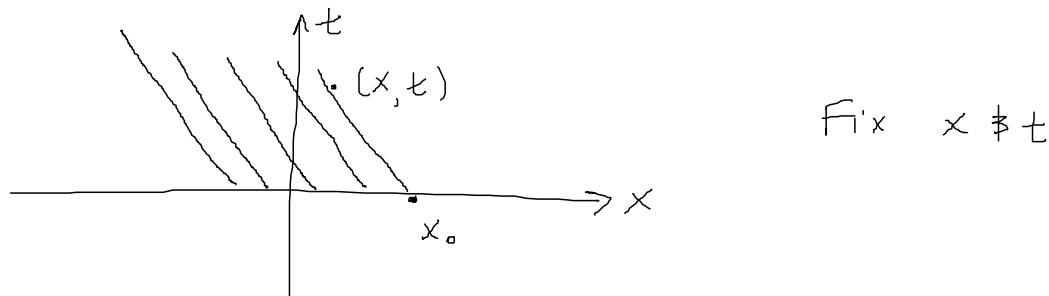
$$u_t + \vec{b} \cdot \nabla u = a(x)$$

in $\mathbb{R} \times (0, \infty)$ with initial data

$$u(x, 0) = g(x).$$

We did this by reducing the problem to an ODE that held on a line

For example $u_t - u_x = a(x, t)$



if you consider the line $\vec{x}(s) = \left(x + \frac{t-s}{a} \right)$ in the $x-t$ plane.

$$\begin{aligned} \text{Then } \frac{d}{ds} u(\vec{x}(s)) &= u_x \frac{dx}{ds} + u_t \frac{dt}{ds} = -u_x(\vec{x}(s)) + u_t(\vec{x}(s)) \\ &= a(\vec{x}(s)) \end{aligned}$$

so on the curve $\vec{x}(s)$ the PDE reduces

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To the ODE

$$\frac{d}{ds} u(\vec{x}(s)) = a(\vec{x}_1(s))$$

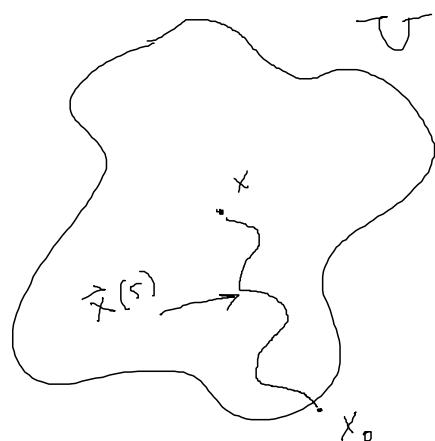
with initial data

$$u(\vec{x}(0)) = g(x_0)$$

We then explicitly solved the ODE, deducing a solution $u(x, t)$.

Idea: given a PDE in \mathcal{V} and $\Gamma \subseteq \partial\mathcal{V}$

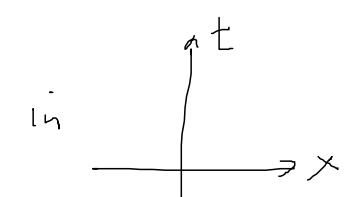
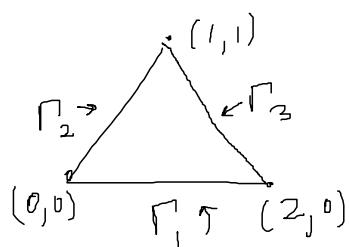
ifnt and boundary data $u = g$ on Γ



Let $x \in \mathcal{V}$. Can I find a curve connecting x to Γ so that the PDE reduces to an ODE on the curve w/ initial data determined by where the curve intersects the boundary?

Also, are there restrictions on the boundary data you can specify?

Consider $u_t - u_x = 0$ on

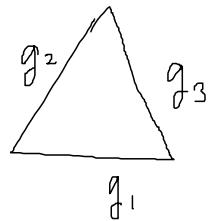


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Since the characteristic running through (x, t) is

$$\vec{x}(s) = \begin{pmatrix} x+t-s \\ s \end{pmatrix}$$

and $u(\vec{x}(s))$ is independent of s , you see that



the values g_1 will be advected to Γ_2 . If g_2 disagrees with the advected values then it's impossible for

$$\lim_{(x,t) \rightarrow \Gamma_2} u(x,t) = g_2 \text{ to hold.}$$

That is, there's no solution $u \in C(\bar{\Omega})$.

What about g_3 ? You quickly realize that if $g_3 \neq \lim_{(x,t) \rightarrow \Gamma_3} g_1(x,0)$ then the solution $u \notin C(\bar{\Omega})$.

So you see that

In order to get $u \in C(\bar{\Omega})$ either you specify the boundary values on $\partial\Omega$ w/ compatibility in mind or you only specify the data on Γ_1 and let the solution in Ω and continuity determine the value of u on $\partial\Omega - \Gamma_1$.

Let's do an example

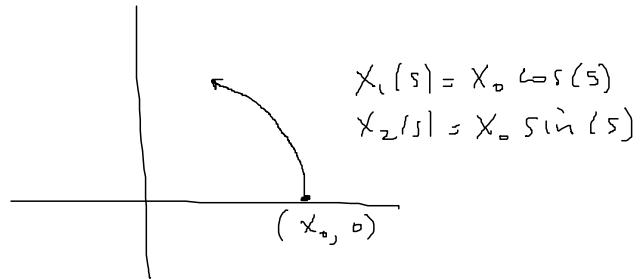
$$\text{Let } U = \{x_1 > 0, x_2 > 0\} \subseteq \mathbb{R}^2$$

$$\Gamma = \{x_1 > 0, x_2 = 0\} \subset \partial U$$

$$\left\{ \begin{array}{l} x_1 \frac{\partial u}{\partial x_2} - x_2 \frac{\partial u}{\partial x_1} = u \quad \text{in } U \\ u = g \quad \text{on } \Gamma \end{array} \right.$$

consider the ODEs

$$\left\{ \begin{array}{l} \frac{dx_1}{ds} = -x_2(s) \\ \frac{dx_2}{ds} = x_1(s) \end{array} \right.$$



$$\begin{aligned} \text{Then } \frac{d}{ds} u(x_1(s), x_2(s)) &= \frac{\partial u}{\partial x_1}(x_1(s), x_2(s)) \frac{dx_1}{ds} + \frac{\partial u}{\partial x_2}(x_1(s), x_2(s)) \frac{dx_2}{ds} \\ &= -x_2(s) \frac{\partial u}{\partial x_1}(x_1(s), x_2(s)) + x_1(s) \frac{\partial u}{\partial x_2}(x_1(s), x_2(s)) \\ &= u(x_1(s), x_2(s)) \end{aligned}$$

so along the curve $(x_1(s), x_2(s))$ u is growing

exponentially



$$\begin{aligned} x_1(s) &= x_0 \cos(s) \\ x_2(s) &= x_0 \sin(s) \end{aligned}$$

$$u(x_1(s), x_2(s)) = u(x_0, 0) e^s = g(x_0) e^s$$

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Okay, we've found $u(x_1(s), x_2(s))$. We want $u(x_1, x_2)$.

Fix x_1, x_2 this will determine x_0 and s .

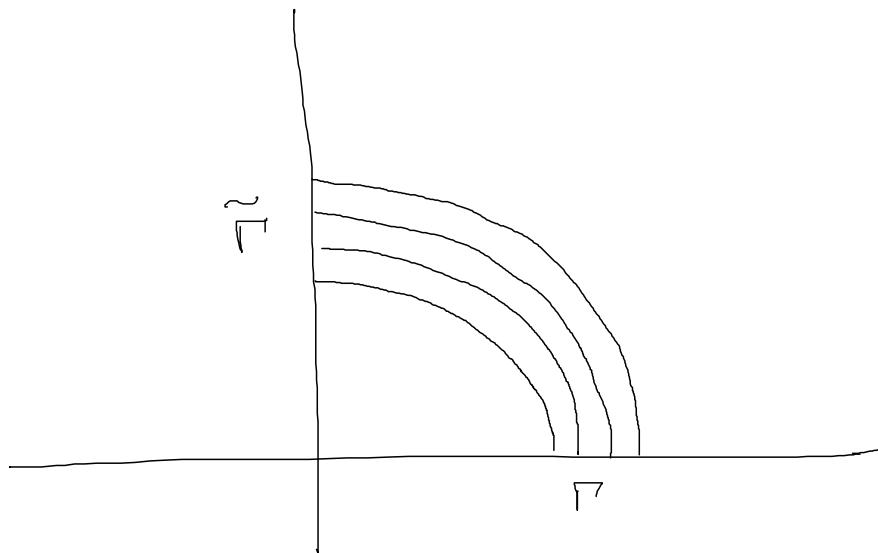
$$\begin{aligned} x_1 &= x_0 \cos(s) \\ x_2 &= x_0 \sin(s) \end{aligned} \Rightarrow \quad x_0 = \sqrt{x_1^2 + x_2^2}$$

$$s = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

\therefore

$$u(x_1, x_2) = g\left(\sqrt{x_1^2 + x_2^2}\right) e^{\tan^{-1}\left(\frac{x_2}{x_1}\right)}$$

check directly that $x_1 \frac{\partial u}{\partial x_2} - x_2 \frac{\partial u}{\partial x_1} = u$ holds in \mathcal{U}



$$\text{fix } x_2 > 0 \quad \lim_{x_1 \rightarrow 0} u(x_1, x_2) = g\left(\sqrt{x_2^2}\right) e^{\frac{\pi i}{2}}$$

$$\lim_{\substack{x \rightarrow \tilde{\Gamma} \\ x_2 \rightarrow 0}} u(x, x_2) \in \mathbb{C}^{\frac{\pi i}{2}} g(0)$$

$$x \rightarrow (0, 0)$$

If $g(0) \neq 0$ then discontinuity at $(0, 0)$

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Okay, now to trying to find those miraculous curves on which we reduce the PDE to an ODE

Will consider first-order PDE:

$$\begin{cases} F(Du, u, x) = 0 & \text{in } \mathcal{U} \subseteq \mathbb{R}^n \\ u = g & \text{on } \partial\mathcal{U} = \Gamma \end{cases}$$

assume g is smooth on $\partial\mathcal{U}$ and

$F(p, z, x)$ is a smooth function on \mathbb{R}^{n+1+n}

Fix $x \in \mathcal{U}$. Want to find $u(x)$ by
computing an ODE initial value problem on
 $\vec{x}(s)$ which is a curve in \mathcal{U} that connects
 $x \neq x_0 \in \Gamma \subseteq \partial\mathcal{U}$

How we choose $\vec{x}(s)$ will depend on the particular
PDE in question.

$$\vec{x}(s) \subseteq \mathbb{R}^n$$

$$\text{Let } z(s) := u(\vec{x}(s)) \in \mathbb{R} \quad \vec{p}(s) := Du(\vec{x}(s)) \in \mathbb{R}^n$$

$$p_i(s) = \frac{\partial u}{\partial x_i}(\vec{x}(s))$$

$$\frac{d}{ds} p_i(s) = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(\vec{x}(s)) \frac{dx_j}{ds}(s)$$

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The PDE $F(Du, u, x) = 0$ has Du embedded in the (poss.) nonlinear function F . We can extricate $u_{x_i x_j}$ by differentiating the PDE.

$$\frac{\partial}{\partial x_i} F(Du, u, x) = \sum_{j=1}^n \frac{\partial F}{\partial p_j} \frac{\partial^2 u}{\partial x_j \partial x_i} + \frac{\partial F}{\partial z} \frac{\partial u}{\partial x_i} + \frac{\partial F}{\partial x_i} = 0$$

Let's be careful w/ the arguments ...

$$\textcircled{*} \quad \sum_{j=1}^n \frac{\partial F}{\partial p_j}(Du, u, x) \frac{\partial^2 u}{\partial x_j \partial x_i}(x) + \frac{\partial F}{\partial z}(Du, u, x) \frac{\partial u}{\partial x_i}(x) + \frac{\partial F}{\partial x_i}(Du, u, x) = 0$$

If u is a solution of the PDE $F(Du, u, x) = 0$ then it's also a solution of the n PDEs $\textcircled{*}$ on T

Recall $\vec{p}(s) := Du(\vec{x}(s))$ $z(s) := u(\vec{x}(s))$ when $\vec{x}(s) \in T$

so we have n PDEs,

$$\begin{aligned} & \sum_{j=1}^n \frac{\partial F}{\partial p_j}(\vec{p}(s), z(s), \vec{x}(s)) \frac{\partial^2 u}{\partial x_j \partial x_i}(\vec{x}(s)) \\ & + \frac{\partial F}{\partial z}(\vec{p}(s), z(s), \vec{x}(s)) \frac{\partial u}{\partial x_i}(\vec{x}(s)) + \frac{\partial F}{\partial x_i}(\vec{p}(s), z(s), \vec{x}(s)) = 0 \end{aligned}$$

⑦

We choose $\vec{x}(s)$ so that the 2nd derivatives on u are taken care of;

$$\text{assume } \frac{d}{ds} x_j = \frac{\partial F}{\partial p_j} (\vec{p}(s), z(s), \vec{x}(s))$$

$$\text{then } \sum_{j=1}^n \frac{\partial F}{\partial p_j} \frac{\partial^2 u}{\partial x_i \partial x_j} x_j = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j} x_j \frac{d x_j}{ds} = \frac{d}{ds} \frac{\partial u}{\partial x_i} (\vec{p}(s), z(s), \vec{x}(s)) \\ = \frac{d}{ds} p_i (\vec{p}(s), z(s), \vec{x}(s))$$

So the n PDEs

$$\sum_{j=1}^n \frac{\partial F}{\partial p_j} \frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial F}{\partial z} p_i (\vec{x}(s)) + \frac{\partial F}{\partial x_i} = 0$$

transform into

$$\frac{d}{ds} p_i (\vec{p}(s), z(s), \vec{x}(s)) = - \frac{\partial F}{\partial z} (\vec{p}(s), z(s), \vec{x}(s)) p_i (\vec{x}(s)) - \frac{\partial F}{\partial x_i} (\vec{p}(s), z(s), \vec{x}(s))$$

We now have n ODEs for the components of $\vec{p}(s)$, we have n ODEs for the components of $\vec{x}(s)$.

We need an ODE for $z(s)$.

$$\begin{aligned}\frac{d}{ds} z(s) &= \frac{d}{ds} u(\vec{x}(s)) \\ &= \sum_{j=1}^n \frac{\partial u}{\partial x_j} (\vec{x}(s)) \frac{dx_j}{ds}(s) \\ &= \sum_{j=1}^n \frac{\partial u}{\partial x_j} (\vec{x}(s)) \frac{\partial F}{\partial p_j} (\vec{p}(s), z(s), \vec{x}(s)) \\ &= \sum_{j=1}^n p_j(s) \frac{\partial F}{\partial p_j} (\vec{p}(s), z(s), \vec{x}(s)).\end{aligned}$$

Our $2n+1$ ODES:

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- a) $\frac{d}{ds} \vec{p}(s) = -D_x F(\vec{p}(s), z(s), \vec{x}(s)) - D_z F(\vec{p}(s), z(s), \vec{x}(s)) \vec{p}(s)$
 - b) $\frac{d}{ds} z(s) = D_p F(\vec{p}(s), z(s), \vec{x}(s)) \cdot \vec{p}(s)$
 - c) $\frac{d}{ds} \vec{x}(s) = D_p F(\vec{p}(s), z(s), \vec{x}(s))$

Given the PDE $F(Du, u, x) = 0$

we have found the $2n+1$ characteristic equations.

$\vec{p}(s)$, $\vec{z}(s)$, $\vec{x}(s)$ are the characteristics

Q: What are the initial conditions for the system of ODEs?

A: patience...

Theorem: Let $u \in C^2(\bar{\Omega})$ solve the nonlinear, first-order PDE $F(Du, u, x) = 0$ in $\bar{\Omega}$,

Assume $\vec{x}(\cdot)$ solves the ODE

$$\frac{d\vec{x}}{ds} = D_p F(\vec{p}(s), \vec{z}(s), \vec{x}(s))$$

where $\vec{p}(s) := Du(\vec{x}(s))$ and $\vec{z}(s) := u(\vec{x}(s))$. Then

$\vec{p}(s)$ and $\vec{z}(s)$ solve

$$\frac{d\vec{p}}{ds} = -D_x F(\vec{p}(s), \vec{z}(s), \vec{x}(s)) - D_{\vec{z}} F(\vec{p}(s), \vec{z}(s), \vec{x}(s)) \vec{p}(s)$$

$$\frac{d\vec{z}}{ds} = D_p F(\vec{p}(s), \vec{z}(s), \vec{x}(s)) \cdot \vec{p}(s).$$

Proof: we did it above.

Time for some examples! (sof!)

ex: $F(Du, u, x) = \vec{b}(x) \cdot Du(x) + c(x)u(x) = 0.$

Then $F(p, z, x) = \vec{b}(x) \cdot \vec{p} + c(x)z$

$$D_p F = \vec{b}(x), \quad D_z F = c(x)$$

so $\frac{d\vec{x}}{ds}(s) = \vec{b}(\vec{x}(s))$

$$\frac{d\vec{z}}{ds} = \vec{b}(\vec{x}(s)) \cdot \vec{p}(\vec{x}(s)) = -c(\vec{x}(s))\vec{z}(s)$$

$\frac{d\vec{p}}{ds}$ = a mess and it'll turn out not to matter.

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