

Energy Methods + the Wave Equation

Uniqueness of solutions to the initial value, boundary value problem:

Theorem: Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set with C^1 boundary $\partial\Omega$ and let

$$\Omega_T := \Omega \times (0, T]$$

$$\Gamma_T := \overline{\Omega_T} - \Omega_T$$

When $T > 0$. Then there exists at most one function $u \in C^2(\overline{\Omega_T})$ solving

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega_T \\ u = g & \text{on } \Gamma_T \\ u_t = h & \text{on } \Omega \times \{t=0\} \end{cases}$$

Proof: Assume \exists two solutions and let $w := u - \tilde{u}$. Then w solves

$$\begin{cases} w_{tt} - \Delta w = 0 & \text{in } \Omega_T \\ w = 0 & \text{on } \Gamma_T \\ w_t = 0 & \text{on } \Omega \times \{t=0\} \end{cases}$$

Let $E(t) := \frac{1}{2} \int_{\Omega} [w_t^2(x, t) + |\nabla w(x, t)|^2] dx$

z

Since $w \in C^2(\overline{U_T})$ $\mathcal{E}(t)$ is differentiable.

$$\frac{d}{dt} \mathcal{E}(t) = \int_U w_t w_{tt} + Dw \cdot Dw_t \, dx$$

$$= \int_U w_t (w_{tt} - \Delta w) \, dx$$

using the divergence theorem &
 $w_t = 0$ on ∂U .

$$= 0$$

Since $w_{tt} - \Delta w = 0$.

Thus $\frac{d}{dt} \mathcal{E}(t) = 0$ on $(0, T)$, since $\mathcal{E}(t)$ is

continuous and $\mathcal{E}(0) = 0$, $\mathcal{E} \equiv 0$ on $[0, T]$

Since w_t and Dw are C^1 we conclude that

$w_t \equiv 0$ and $Dw \equiv 0$. Since $w = 0$ at $t = 0$ we

then conclude that $w \equiv 0$ on $\overline{U_T}$, as claimed. //

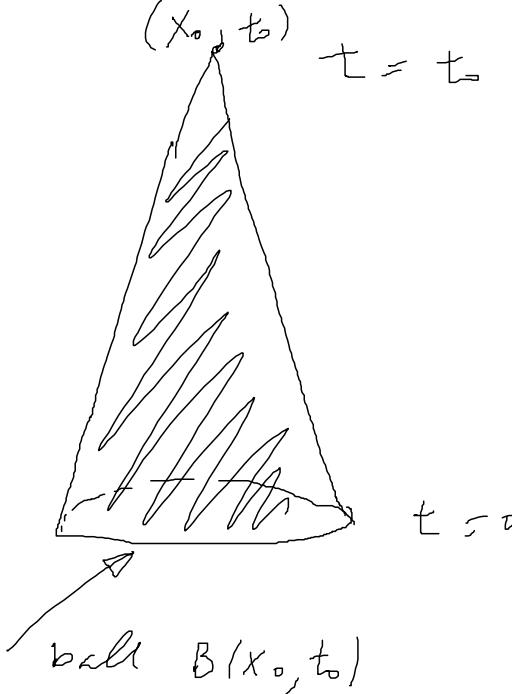
Domain of Dependence

Fix $x_0 \in \mathbb{R}^n$ and $t_0 > 0$ and consider

$$C = \left\{ (x, t) \mid t < t_0 \text{ and } |x - x_0| \leq t_0 - t \right\}$$

This is a cone in $\mathbb{R}^n \times [0, \infty)$ with tip at (x_0, t_0)

(3)



the shaded cone is the
set C .

Theorem: (Finite speed of propagation)

If $u=u(t=0)$ on $B(x_0, t_0) \times \{t=0\}$ then
 $u \equiv 0$ in C .

Specifically, a perturbation outside $B(x_0, t_0) \times \{t=0\}$ cannot influence $u(x_0, t_0)$. We already know this for solutions on \mathbb{R}^n . Just by looking at the exact solution. However, this proof won't require exact solution. And it holds on $\mathcal{U} \subseteq \mathbb{R}^n$ as long as $B(x_0, t_0) \subset \mathcal{U}$.

Proof: Let

$$\mathcal{E}(t) := \frac{1}{2} \int_{B(x_0, t_0-t)} |u_t|^2 + |\nabla u|^2 dx$$

(4)

$$\frac{d}{dt} \mathcal{E} = \int_{B(x_0, t_0-t)} u_t u_{tt} + Du \cdot D u_t \, dx$$

$$B(x_0, t_0-t)$$

$$= \frac{1}{2} \int_{\partial B(x_0, t_0-t)} u_t^2 + |Du|^2 \, dS(x)$$

$$\partial B(x_0, t_0-t)$$

$$\left(\text{since } \frac{d}{dr} \int_{B(x_0, r)} f(y) dy = \int_{\partial B(x_0, r)} f(y) dy \right)$$

$$= \int_{B(x_0, t_0-t)} u_t (u_{tt} - \Delta u) dx + \int_{\partial B(x_0, t_0-t)} u_t \frac{\partial u}{\partial \nu} \, dS(x)$$

$$- \frac{1}{2} \int_{\partial B(x_0, t_0-t)} u_t^2 + |Du|^2 \, dS(x)$$

$$= \int_{\partial B(x_0, t_0-t)} u_t \frac{\partial u}{\partial \nu} - \frac{1}{2} u_t^2 - \frac{1}{2} |Du|^2 \, dS(x)$$

Young's inequality

$$\left| u_t \frac{\partial u}{\partial \nu} \right| = \left| u_t \right| \left| \frac{\partial u}{\partial \nu} \right| \leq \frac{1}{2} |u_t|^2 + \frac{1}{2} \left| \frac{\partial u}{\partial \nu} \right|^2$$

$$\leq \frac{1}{2} u_t^2 + \frac{1}{2} |Du|^2$$

$$\Rightarrow \frac{d\mathcal{E}}{dt} \leq 0, \Rightarrow \mathcal{E}(t) \leq \mathcal{E}(0) \leq 0 \quad \text{for all } t \in [0, t_0]$$

$\Rightarrow u \equiv 0 \text{ on } C, \text{ as desired. } //$