

Solving $U_{tt} - \Delta U = 0$ in \mathbb{R}^3

$$\begin{cases} U_{tt} - \Delta U = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ U = g, \quad U_t = h & \text{on } \mathbb{R}^3 \times \{t=0\} \end{cases}$$

will solve this via spherical means and the Euler-Poisson-Darboux system

$$\textcircled{*} \quad \begin{cases} U_{tt} - U_{rr} - \frac{2}{r} U_r = 0 & \text{in } (0, \infty) \times (0, \infty) \\ U = G, \quad U_t = H & \text{in } [0, \infty) \times \{t=0\} \end{cases}$$

where

$$U(x; r, t) := \int\limits_{\partial B(x, r)} u(y, t) dy$$

$$G(x; r) := \int\limits_{\partial B(x, r)} g(y) dy \quad H(x; r) := \int\limits_{\partial B(x, r)} h(y) dy$$

To solve $\textcircled{*}$ we introduce

$$\tilde{U} := rU, \quad \tilde{G} := rG, \quad \tilde{H} := rH$$

$$\begin{aligned} \tilde{U}_{tt} &= rU_{tt} = r \left[U_{rr} + \frac{2}{r} U_r \right] \\ &= rU_{rr} + 2U_r \\ &= (U + rU_r)_r \\ &= (rU)_{rr} = \tilde{U}_{rr} \end{aligned}$$

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and so

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ \tilde{U} = \tilde{G}, \tilde{U}_t = \tilde{H} & \text{on } \mathbb{R}_+ \times \{t=0\} \\ \tilde{U}(0, t) = 0 & \forall t \end{cases}$$

this is the wave equation on the half line w/
reflecting boundary conditions. we know how
to solve it explicitly:

for $0 \leq r \leq t$

$$\begin{aligned} \tilde{U}(x; r, t) = & \frac{1}{2} [\tilde{G}(x; r+t) - \tilde{G}(x; t-r)] \\ & + \frac{1}{2} \int_{-r+t}^{r+t} \tilde{H}(x; \rho) d\rho \end{aligned}$$

Since $\lim_{r \rightarrow 0} \tilde{U}(x; r, t) = u(x, t)$

we see $\lim_{r \rightarrow 0} \frac{\tilde{U}(x; r, t)}{r} = u(x, t)$

$$\begin{aligned} \Rightarrow u(x, t) &= \lim_{r \rightarrow 0} \frac{1}{2r} [\tilde{G}(x; r+t) - \tilde{G}(x; t-r)] \\ &\quad + \frac{1}{2r} \int_{-r+t}^{r+t} \tilde{H}(x; \rho) d\rho \\ &= \tilde{G}'(x; t) + \tilde{H}(x; t) \\ &= \frac{\partial}{\partial t} \left[t \int_{\partial B(x, t)} g(y) ds(y) \right] + t \int_{\partial B(x, t)} h(y) ds(y) \end{aligned}$$

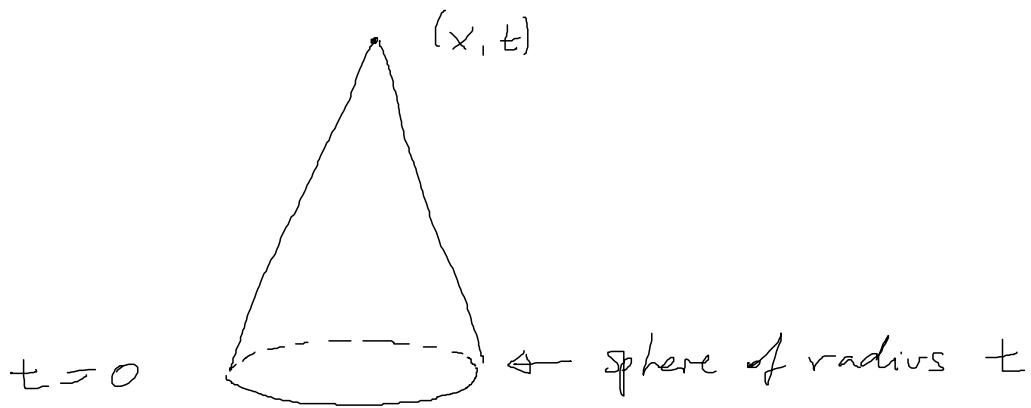
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$$\begin{aligned}
& \frac{\partial}{\partial t} \left[t \int_{\partial B(x,t)} g(y) ds(y) \right] \\
&= \int_{\partial B(x,t)} g(y) ds(y) + t \frac{\partial}{\partial t} \int_{\partial B(x,t)} g(y) ds(y) \\
&= \int_{\partial B(x,t)} g(y) ds(y) + t \frac{\partial}{\partial t} \int_{\partial B(0,1)} g(x+tz) ds(z) \\
&= \int_{\partial B(x,t)} g(y) ds(y) + t \int_{\partial B(0,1)} Dg(x+tz) \cdot z ds(z) \\
&= \int_{\partial B(x,t)} g(y) ds(y) + t \int_{\partial B(x,t)} Dg(y) \cdot \frac{y-x}{t} ds(y) \\
&= \int_{\partial B(x,t)} g(y) ds(y) + \int_{\partial B(x,t)} Dg(y) \cdot (y-x) ds(y)
\end{aligned}$$

Hence the solution of the wave equation in 3d is

$$u(x,t) = \int_{\partial B(x,t)} \theta(y) + g(y) + Dg(y) \cdot (y-x) ds(y)$$

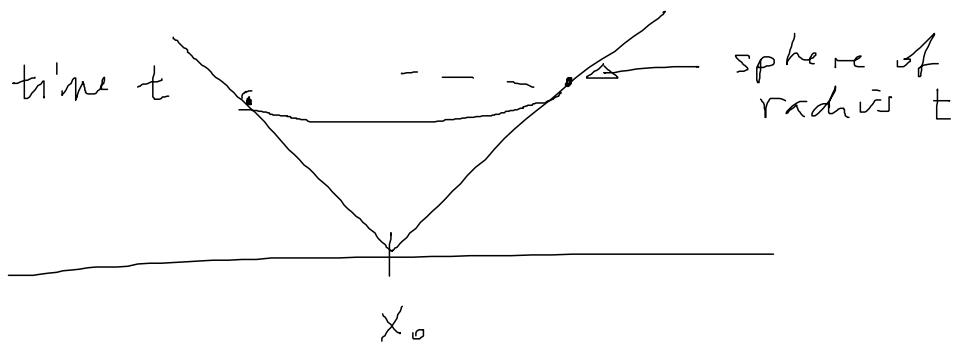
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So a solution at time t is determined by the initial data on a sphere of radius t .

"the domain of dependence for $u(x, t)$ is $\partial B(x, t)$."

Alternatively, can ask what points (x, t) are influenced by initial data at x_0 ...



If you're standing a distance r from x_0 then you'll hear the signal at time r . And then the sound will be gone. (cone of influence)

the solution in 3d is called "Kirchhoff's Formula".

Now we'll solve $u_{tt} - \Delta u = 0$ in \mathbb{R}^3 .

$$\left\{ \begin{array}{l} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^3 \times (0, \infty) \\ u = g, \quad u_t = h \quad \text{on } \mathbb{R}^3 \times \{t=0\} \end{array} \right.$$

To do this, we'll do something a little cheezy... or clever  Since we've already solved the wave eqn in \mathbb{R}^3 , we'll seek solutions that are independent of x_3 .

$$\bar{u}(x_1, x_2, x_3, t) := u(x_1, x_2, t)$$

$$\begin{aligned} \text{then } \bar{u}_{tt} - \bar{u}_{x_1 x_1} - \bar{u}_{x_2 x_2} - \bar{u}_{x_3 x_3} &= u_{tt} - u_{x_1 x_1} - u_{x_2 x_2} \\ &= u_{tt} - \Delta u = 0 \end{aligned}$$

$$\bar{g}(x_1, x_2, x_3) := g(x_1, x_2)$$

$$\bar{h}(x_1, x_2, x_3) := h(x_1, x_2)$$

solve $\left\{ \begin{array}{l} \bar{u}_{tt} - \Delta \bar{u} = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty) \\ \bar{u} = \bar{g}, \quad \bar{u}_t = \bar{h} \quad \text{on } \mathbb{R}^3 \times \{t=0\} \end{array} \right.$

$$\text{let } x = (x_1, x_2) \in \mathbb{R}^2$$

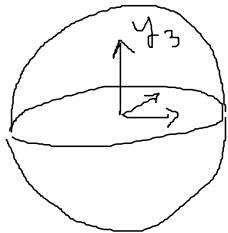
$$\bar{x} = (x_1, x_2, 0) \in \mathbb{R}^3$$

Kirchhoff's formula for solutions in \mathbb{R}^3

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$$\bar{u}(\bar{x}, t) = \frac{\partial}{\partial t} \left(t \int_{\partial B(\bar{x}, t)} \bar{g}(y) dS(y) \right) + t \int_{\partial B(\bar{x}, t)} \bar{h}(y) dS(y)$$

now $\int_{\partial B(\bar{x}, t)} \bar{g}(y) dS(y) = \frac{1}{4\pi t^2} \int_{\partial B(\bar{x}, t)} g(y) dS(y)$



$$\bar{g}(y_1, y_2, y_3) = g(y_1, y_2)$$

upper half of ball:

$$y_3 = \gamma(y) = \sqrt{t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2} = \sqrt{t^2 - |y - x|^2}$$

bottom half of ball:

$$y_3 = -\gamma(y)$$

$$\Rightarrow \frac{1}{4\pi t^2} \int_{\partial B(\bar{x}, t)} \bar{g}(y) dS(y) = \frac{2}{4\pi t^2} \int_{B(x, t)} g(y) \sqrt{1 + |\nabla \gamma(y)|^2} dy$$

(the factor of 2 comes from there being two hemispheres.)

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$$\text{Now } \gamma(y) = (t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2)^{1/2}$$

$$\frac{\partial \gamma}{\partial y_1} = \frac{-(y_1 - x_1)}{\gamma(y)} \quad \frac{\partial \gamma}{\partial y_2} = \frac{-(y_2 - x_2)}{\gamma(y)}$$

$$\sqrt{1 + |\nabla \gamma(y)|^2} = \sqrt{\gamma(y)^2 + (y_1 - x_1)^2 + (y_2 - x_2)^2} \frac{1}{\gamma(y)} = \frac{t}{\gamma(y)}$$

$$\begin{aligned} \oint_{\partial B(\bar{x}, t)} \bar{g}(y) dS(y) &= \frac{1}{2\pi t} \int_{B(x, t)} \frac{g(y)}{\gamma(y)} dy \\ &= \frac{t}{2} \int_{B(x, t)} \frac{g(y)}{\sqrt{t^2 - |x-y|^2}} dy \end{aligned}$$

hence

$$u(x, t) - \bar{u}(\bar{x}, t) = \frac{\partial}{\partial t} \left(\frac{t^2}{2} \int_{B(x, t)} \frac{g(y)}{\sqrt{t^2 - |x-y|^2}} dy \right) + \frac{t^2}{2} \int_{B(x, t)} \frac{h(y)}{\sqrt{t^2 - |x-y|^2}} dy$$

Now to unravel that time derivative ...

$$\frac{\partial}{\partial t} \left(\frac{t^2}{2} \int_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |x-y|^2}} dy \right) = \frac{\partial}{\partial t} \left(\frac{1}{2\pi} \int_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |x-y|^2}} dy \right)$$

$$= \frac{\partial}{\partial t} \left(\frac{1}{2\pi} \int_{B(0,1)} \frac{g(x+tz)}{t\sqrt{1-|z|^2}} t^2 dz \right)$$

$$= \frac{\partial}{\partial t} \left(\frac{t}{2\pi} \int_{B(0,1)} \frac{g(x+tz)}{\sqrt{1-|z|^2}} dz \right)$$

$$= \frac{1}{2\pi} \int_{B(0,1)} \frac{g(x+tz)}{\sqrt{1-|z|^2}} dz + \frac{t}{2\pi} \int_{B(0,1)} \frac{Dg(x+tz) \cdot z}{\sqrt{1-|z|^2}} dz$$

$$= \frac{t}{2} \int_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |x-y|^2}} dy + \frac{t}{2} \int_{B(x,t)} \frac{Dg(y) \cdot (y-x)}{\sqrt{t^2 - |x-y|^2}} dy$$

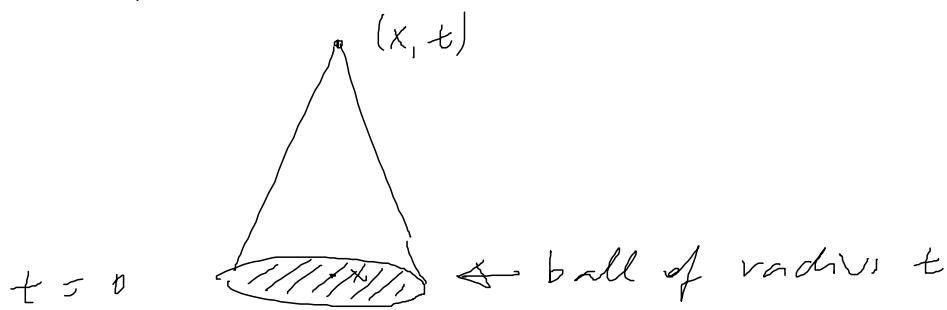
putting this back into the formula for $u(x,t)$

$$u(x,t) = \frac{1}{2} \int_{B(x,t)} \frac{t^2 h(y) + t g(y) + t Dg(y) \cdot (y-x)}{\sqrt{t^2 - |y-x|^2}} dy$$

this is Poisson's formula for the solution of
the wave equation IVP in \mathbb{R}^2

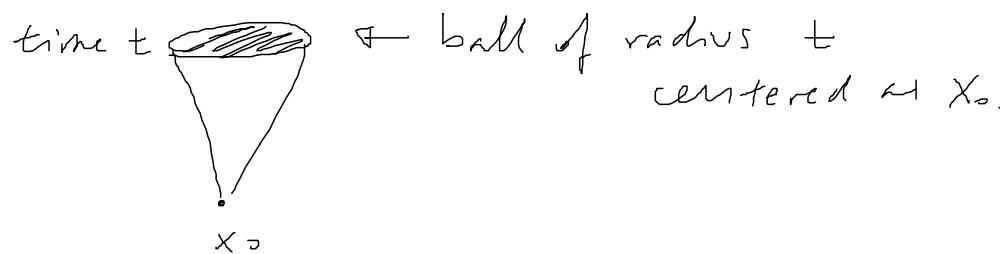
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Note that



The solution $u(x, t)$ is determined by the initial data in a ball $B(x, t)$.

given a signal at x_0 , what points (x, t) will it influence?



If you're located at a point r away from x_0
then at time $t=r$ you'll hear the signal at x_0 .

And you'll keep hearing it! Vibes!

What happened? We solved the \mathbb{R}^2 problem by going to \mathbb{R}^3 . A sphere in \mathbb{R}^3 projects to a ball in \mathbb{R}^2 . A hollow cone in $\mathbb{R}^3 \times (0, \infty)$ projects to a solid cone in $\mathbb{R}^2 \times (0, \infty)$. It was unavoidable.

Fact: \mathbb{R}^n if $n \geq 1$, n odd then soln like \mathbb{R}^3 soln
if n even then soln like \mathbb{R}^2 soln.