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The Wave Equation

The wave equation is

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^{n+1}$$

The inhomogeneous wave equation is

$$u_{tt} - \Delta u = f$$

Subject to appropriate initial + boundary conditions

Note: we'll find that unlike the heat equation, the wave equation has finite speed of propagation. Furthermore, it doesn't smooth the initial data.

We'll analyze the wave equation for dimensions 1, 2, and 3. The book does general \mathbb{R}^n but it's more complicated than needed at first.

1-d. Consider the initial value problem

on \mathbb{R} :

$$\left\{ \begin{array}{ll} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t=0\} \\ u_t = h & \text{on } \mathbb{R} \times \{t=0\} \end{array} \right.$$

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In \mathbb{R}^1 , the operator $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$
can be factored:

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right)$$

this will allow us to solve the wave equation
explicitly

$$\text{let } v(x, t) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) u$$

Since u

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) u(x, t) &= 0 \\ \Rightarrow \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) v(x, t) &= 0 \quad \Rightarrow \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 0 \end{aligned}$$

$$\Rightarrow v(x, t) = \alpha(x-t) \quad \text{for some function } \alpha$$

$$\Rightarrow \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) u(x, t) = \alpha(x-t)$$

$$\Rightarrow \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = \alpha(x-t) \quad \text{can solve this.}$$

$$u(x, t) = b(x+t) + \int_0^t \alpha(x+(t-s)-s) ds$$

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$$\Rightarrow u(x, t) = b(x+t) + \frac{1}{2} \int_{x-t}^{x+t} a(y) dy$$

Now use the initial data to determine the unknown functions a and b .

$$u(x, 0) = b(x+0) + \frac{1}{2} \int_{x-0}^{x+0} a(y) dy \Rightarrow u(x, 0) = b(x) = g$$

$$\therefore u(x, t) = g(x+t) + \frac{1}{2} \int_{x-t}^{x+t} a(y) dy$$

$$\frac{\partial u}{\partial t} = g'(x+t) + \frac{1}{2} [a(x+t) + a(x-t)]$$

$$\Rightarrow \frac{\partial u}{\partial t}(x, 0) = g'(x) + a(x) = h(x)$$

$$\Rightarrow a(x) = h(x) - g'(x)$$

$$u(x, t) = g(x+t) + \frac{1}{2} \int_{x-t}^{x+t} h(y) - g'(y) dy$$

$$u(x, t) = \frac{1}{2} [g(x+t) + g(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy$$

This is D'Alembert's formula

Note that

$u(x,t)$ can be written as

$$u(x,t) = F(x+t) + G(x-t)$$

for appropriately chosen F & G . So this tells us that any solution of $u_{tt} - u_{xx} = 0$ w/ u and u_t specified at $t=0$ can be written as the sum of a right-going wave and a left-going wave.

Theorem: Assume $g \in C^k(\mathbb{R})$ and $h \in C^{k-1}(\mathbb{R})$ where $k \geq 2$ and $u(x,t)$ is as defined by d'Alembert's formula. Then

$$\textcircled{1} \quad u \in C^k(\mathbb{R} \times [0, \infty))$$

$$\textcircled{2} \quad u_{tt} - u_{xx} = 0 \quad \text{on } \mathbb{R} \times (0, \infty)$$

$$\textcircled{3} \quad \lim_{\substack{(x,t) \rightarrow (x_0, 0) \\ t > 0}} u(x,t) = g(x)$$

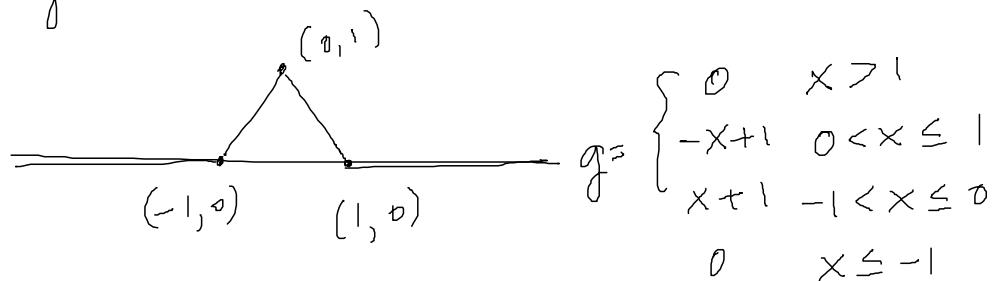
$$\textcircled{4} \quad \lim_{\substack{(x,t) \rightarrow (x_0, 0) \\ t > 0}} u_t(x,t) = h(x).$$

Proof: do it yourself! 😊

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Let's consider a specific solution, the "plucked string"

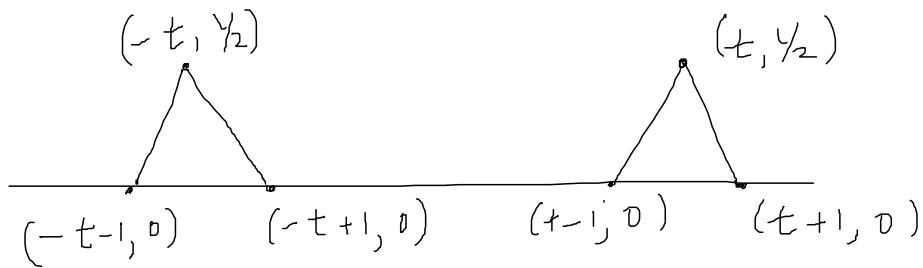
$$u(x, 0) :$$



$$u_t(x, 0) = 0$$

then

$$u(x, t) = \frac{1}{2} [g(x+t) + g(x-t)]$$



two pulses moving apart w/ fixed height.

Notice that the solution is not becoming more smooth as time passes.

Q: What if it's not on \mathbb{R} ? What if there's a boundary?

$$\text{Ex: } \quad \begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ \textcircled{*} \quad \begin{cases} u=g, u_t=h & \text{in } \mathbb{R}_+ \times \{t=0\} \\ u=0 & \text{on } \{x=0\} \times (0, \infty) \end{cases} \end{cases}$$

i.e. a vibrating string where one end is held fixed. In this case, we expect reflection back from the fixed point.

We will solve $\textcircled{*}$ by extending g and h and u to \mathbb{R} as odd functions and then solving on the line.

$$\tilde{u}(x, t) := \begin{cases} u(x, t) & x \geq 0 \\ -u(-x, t) & x < 0 \end{cases}$$

$$\tilde{h}(x) := \begin{cases} h(x) & x \geq 0 \\ -h(-x) & x < 0 \end{cases}$$

$$\tilde{g}(x) := \begin{cases} g(x) & x \geq 0 \\ -h(-x) & x < 0 \end{cases}$$

$$\tilde{u}_{tt} = \begin{cases} u_{tt}(x, t) & x \geq 0 \\ -u_{tt}(-x, t) & x < 0 \end{cases} \Rightarrow u_{tt} - u_{xx} = 0 \quad \text{in } \mathbb{R}_+ \times (0, \infty)$$

$$\tilde{u}_{xx} = \begin{cases} u_{xx}(x, t) & x \geq 0 \\ -u_{xx}(-x, t) & x < 0 \end{cases} \Rightarrow \tilde{u}_{tt} - \tilde{u}_{xx} = 0 \quad \text{in } \mathbb{R}_+ \times (0, \infty)$$



$$\Rightarrow \tilde{u}(x,t) = \frac{1}{2} [\tilde{g}(x+t) + \tilde{g}(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} \tilde{h}(y) dy$$

Case 1: $t \leq x \Rightarrow x-t \geq 0$ no reflection yet

Case 2: $t > x \Rightarrow x-t < 0 \Rightarrow$ reflection

Case 1: $t \leq x$

$$u(x,t) = \frac{1}{2} [\tilde{g}(x+t) + \tilde{g}(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy$$

Case 2: $t > x$

$$u(x,t) = \frac{1}{2} [\tilde{g}(x+t) - \tilde{g}(t-x)] + \frac{1}{2} \int_{x-t}^{x+t} \tilde{h}(y) dy$$

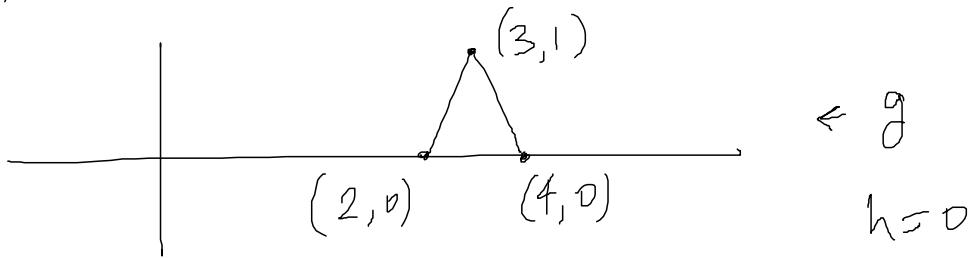
$$= \frac{1}{2} [\tilde{g}(x+t) - \tilde{g}(t-x)] + \frac{1}{2} \int_0^{x+t} h(y) dy$$

$$+ \frac{1}{2} \int_{x-t}^0 h(-y) dy$$

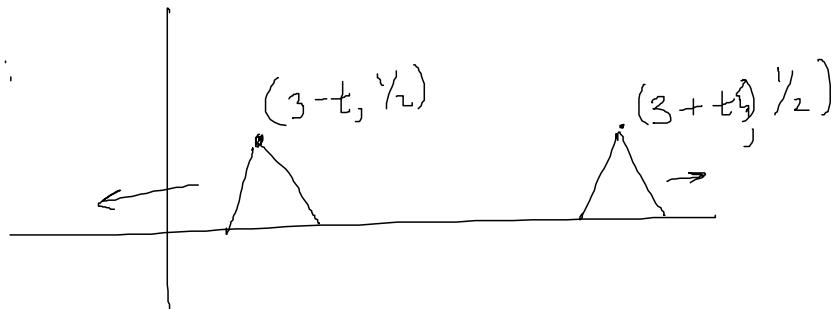
$$= \frac{1}{2} [\tilde{g}(x+t) - \tilde{g}(t-x)] + \frac{1}{2} \int_{t-x}^{x+t} h(y) dy$$

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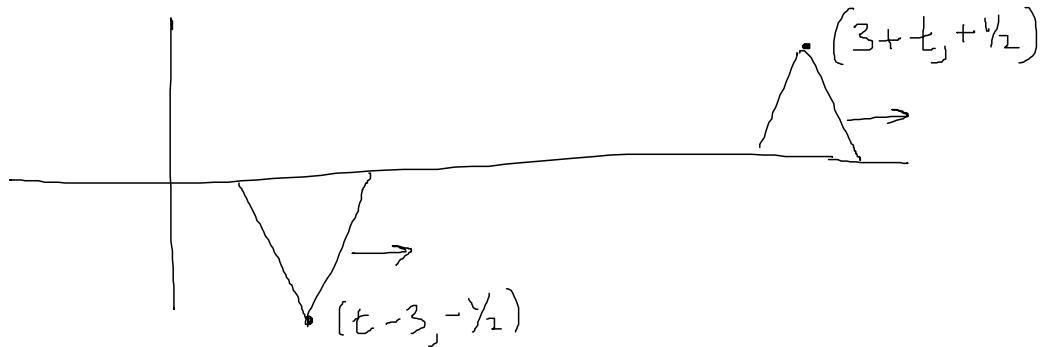
try plucked string initial data again



for $t < 2$:



for $t > 4$



for $2 \leq t \leq 4$ try it your self!

Also, try solving

$$\left\{ \begin{array}{l} u_{tt} - u_{xx} = 0 \quad \text{in } \mathbb{R}_+ \times (0, \infty) \\ u = g, \quad u_t = h \quad \text{on } \mathbb{R}_+ \times \{t=0\} \\ u_x(0, t) = 0 \quad \text{for all } t > 0. \end{array} \right.$$

Now onto higher dimensions.

We'll find that the solutions behave very differently in even + odd dimensions.

Step 1: Spherical Means.

Let $x \in \mathbb{R}^n$, $t > 0$, $r > 0$

$$\begin{aligned} U(x, t, r) &:= \int_{\partial B(x, r)} u(y, t) dS(y) \\ &= \frac{1}{n \omega(n) r^{n-1}} \int_{\partial B(x, r)} u(y, t) dS(y) \end{aligned}$$

The average of $u(x, t)$ over the sphere $\partial B(x, r)$

Similarly,

$$G(x, r) := \int_{\partial B(x, r)} g(y) dS(y)$$

$$H(x, r) := \int_{\partial B(x, r)} h(y) dS(y)$$

Lemma: (Envelop-Poisson-Darboux Equation) Let $n \geq 2$ $m \geq 2$

Fix $x \in \mathbb{R}^n$ and let $u \in C^m(\mathbb{R}^n \times [0, \infty))$

$$u_{tt} + \Delta u = 0$$

then $V \in C^m([0, \infty) \times [0, \infty))$ and

$$\begin{cases} V_{tt} - V_{rr} - \frac{n-1}{r} V_r = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ V = G, \quad V_t = H & \text{on } \mathbb{R}_+ \times \{t=0\} \end{cases}$$

Proof:

$$V(x; r, t) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x, r)} u(y, t) dS(y)$$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(0, 1)} u(x + rz) dS(z)$$

$$\Rightarrow V_r = \frac{1}{n\alpha(n)} \int_{\partial B(0, 1)} Du(x + rz) \cdot z dS(z)$$

$$= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x, r)} Du(y) \cdot \frac{y-x}{r} dS(y)$$

$$= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x, r)} \frac{\partial u}{\partial v}(y) dS(y)$$

$$= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y,t) dy$$

$$U_r = \frac{r}{n} \int_{B(x,r)} \Delta u(y,t) dy$$

Since $u \in C^m$ and $m \geq 2$ we see

$$\lim_{r \rightarrow 0^+} \frac{r}{n} \int_{B(x,r)} \Delta u(y,t) dy = 0$$

$$\Rightarrow \lim_{r \rightarrow 0^+} U_r(x; r, t) = 0$$

and so we know $U_r \in C([0, \infty) \times [0, \infty))$

$$U_r(x; r, t) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y, t) dy$$

$$\text{recall } \frac{d}{dr} \left(\int_{B(x,r)} f(y) dy \right) = \int_{\partial B(x,r)} f(y) dS(y)$$

$$\begin{aligned} \therefore U_{rr}(x; r, t) &= \frac{1-n}{n} \frac{1}{\alpha(n)r^n} \int_{B(x,r)} \Delta u(y, t) dy + \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \Delta u(y, t) dy \\ &= \frac{1-n}{n} \int_{B(x,r)} \Delta u(y, t) dy + \int_{\partial B(x,r)} \Delta u(y, t) dy \end{aligned}$$

$$\text{So } \lim_{r \rightarrow 0} U_{rr} = \left(\frac{1-n}{n} + 1 \right) \Delta u(x, t) = \frac{1}{n} \Delta u(x, t)$$

and so we know $U_{rr} \in C^{\lfloor r \rfloor, \lfloor t \rfloor}([0, \infty) \times [0, \infty))$

proceeding in this way,

$$U \in C^m([0, \infty) \times [0, \infty))$$

Now to find the PDE U satisfies

$$\text{Since } U_r = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y, t) dy \text{ and } u_{tt} = \Delta u$$

$$\text{we have } U_r = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} u_{tt}(y, t) dy$$

$$\Rightarrow r^{n-1} U_r = \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt}(y, t) dy$$

$$\Rightarrow (r^n U_r)_r = \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt}(y, t) dy$$

$$= \frac{r^{n-1}}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} u_{tt}(y, t) dy$$

$$= r^{n-1} U_{tt}(x, r, t)$$

$$\Rightarrow r^{n-1} U_{tt} - (r^{n-1} U_r)_r = 0$$

$$\Rightarrow U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0$$

and so $U(x; r, t)$ satisfies the PDE as claimed.

It remains to show that $U(x; r, t)$ and $U_t(x; r, t)$ achieve the spherically averaged initial data. You should check this on your own



Note that we could have predicted the PDE for U as follows:

$$U_{tt} + \Delta_x u = 0$$

write u in polar coordinates, centered around x_0 ,

then $\Delta_x = \Delta_r + \Delta_\theta$

\uparrow ↗ angular laplacian
radial laplacian

$$= \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r}$$

$U(x; r, t)$ has no angular dependence so $U_{tt} - \Delta_r U = 0$