

More characteristics!

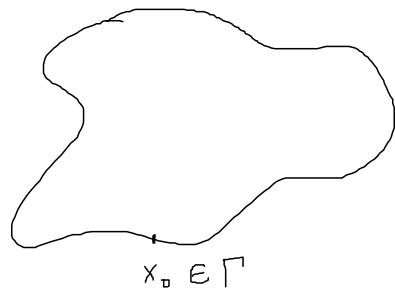
Okay, we've seen how to go from

$$F(\partial u, u, \vec{x}) = 0 \quad \text{on } \Gamma$$

to a system of 2n+1 ODES. How is it that we can be sure that we can go in the other direction?

That is: if we solve the 2n+1 ODES how do we know we can construct a solution (at least locally) in \mathcal{V} ? Basically, this is going to boil down to whether or not the characteristics "fill space" in some way.

Goal:



- 1) solve the 2n+1 ODES starting at $y \in \Gamma \cap (\text{nbhd of } x_0)$
- 2) use the characteristics to define u in $\mathcal{V} \cap (\text{nbhd of } x_0)$
- 3) argue that u solves the PDE.

Note: we aren't arguing that we've got a solution in all of \mathcal{V} ; we're arguing we have a local solution.

Step 0: Straightening the boundary. Let $\vec{x}_0 \in \Gamma$. We want to "straighten" the boundary near \vec{x}_0 .

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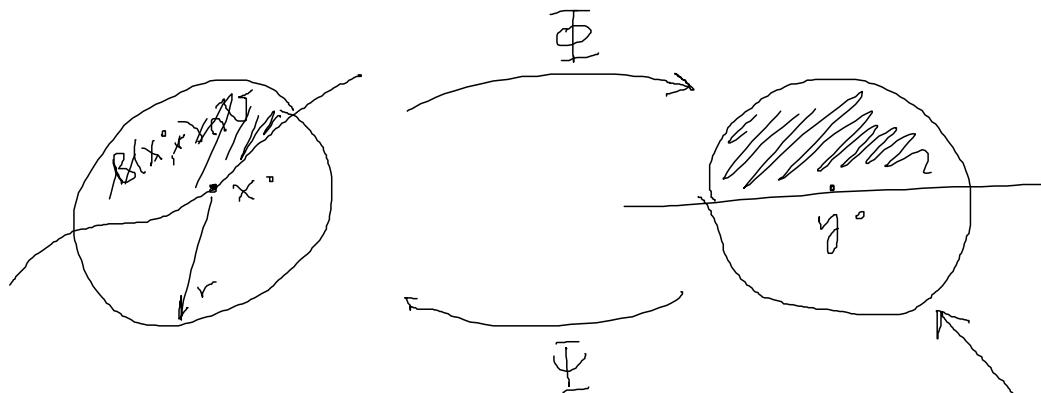
definition: we say $\partial U \cup C^1$ if for each point $x^0 \in \partial U$ $\exists r > 0$ and a C^1 function $\gamma : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ such that - upon relabelling & reorienting the coordinate axes if necessary - we have

$$U \cap B(x_0, r) = \{x \in B(x_0, r) \mid x_n > \gamma(x_1, \dots, x_{n-1})\}$$

that is in a neighborhood of x_0 , ∂U is the graph of a C^1 function γ :

$$\partial U \cap B(x^0, r) = \{ \mid x_n = \gamma(x_1, \dots, x_{n-1}) \}$$

We use Φ to straighten the boundary



define Φ as follows

$$\Phi \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n - \gamma(x_1, x_2, \dots, x_{n-1}) \end{pmatrix}$$

then Φ sends $B(x^0, r) \cap U$ to the shaded region
 Φ sends $B(x^0, r) \cap \partial U$ to the line segment

(3)

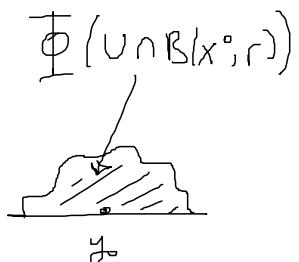
$$\underline{\Psi} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n + \varphi(y_1, y_2, \dots, y_{n-1}) \end{pmatrix}$$

$$\underline{\Phi} = \underline{\Psi}^{-1} \quad \text{and} \quad \det D\underline{\Phi} = \det D\underline{\Psi} = 1$$

$\underline{\Phi}$ is as smooth as φ is.

Consider $F(Du, u, x) = 0$ in $T \cap B(x^*, r)$

$$\text{let } V = \underline{\Phi}(T \cap B(x^*, r))$$



for $y \in \underline{\Phi}(T \cap B(x^*, r))$ let $v(y) := u(\underline{\Phi}(y))$

$$\text{and } u(x) = v(\underline{\Phi}(x)).$$

What PDE does v satisfy?

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^n \frac{\partial v}{\partial y_j} (\underline{\Phi}(x)) \frac{\partial}{\partial x_i} \underline{\Phi}^j(x) \quad x = \underline{\Phi}(y)$$

$$\text{i.e. } D_u(x) = Dv(\underline{\Phi}(x)) D\underline{\Phi}(x)$$

so $F(Du, u, x) = 0$ in $T \cap B(x^*, r)$

$$\Rightarrow F(Dv(y) D\underline{\Phi}(\underline{\Psi}(y)), v(y), \underline{\Psi}(y)) = 0 \quad \text{in } \underline{\Phi}(T \cap B(x^*, r))$$

$$= G(Dv(y), v(y), y) \quad \text{for } G \text{ well chosen.}$$

(G will involve $\underline{\Psi}, D\underline{\Phi}$ etc.)

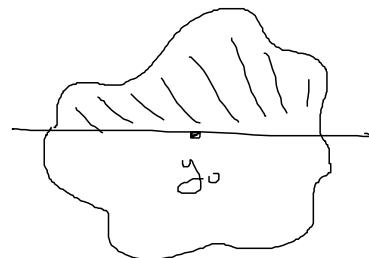
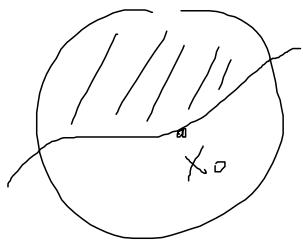
(4)

further $u(x) = g(x)$ on Γ

transforms to $v(y) = g(\psi(y)) = h(y)$

on $= \Phi(\Gamma \cap B(x^*, r))$

To sum up:



solve

$$\begin{cases} F(Du, u, x) = 0 \text{ in shaded region} \\ u = g \text{ on } \Gamma \cap \text{shaded region} \end{cases}$$

solve

$$\begin{cases} G(Dv, v, y) = 0 \text{ in shaded region} \\ v = h \text{ on } \tilde{\Gamma} \end{cases}$$

Note that these two problems look the same! Same structure, it suffices to solve \oplus in which case we know precisely what $\tilde{\Gamma}$ looks like. (It's flat.)

WLOG, consider $\begin{cases} F(Du, u, x) = 0 \text{ in } \Gamma = \overline{\text{shaded region}} \\ u = g \text{ on } \underline{x_0} \subset \{x_n = 0\} \subseteq \mathbb{R}^n \end{cases}$

Consider the characteristic

ODEs $\frac{d\vec{p}}{ds} = \dots \quad \frac{d\vec{z}}{ds} = \dots \quad \frac{d\vec{x}}{ds} = \dots$

We need initial data. $\vec{x}(0) = x^0 \quad \vec{z}(0) = g(x^0) = z^0$

$p^0 := \vec{p}(0) = ??$

We know $\vec{p}(s) = D u(\vec{x}(s))$.

This will give us the first $n-1$ components of $\vec{p}(\cdot)$.

$$(p^*)_i = \frac{\partial g}{\partial x_i}(x^*) \quad i=1 \dots n-1$$

What about the n^{th} component? Deduce it from

$$F(p^*, z^*, x^*) = 0.$$

Initial data: $z^* = g(x^*) \quad \textcircled{P}$

$$\begin{aligned} p^* \text{ solves } & \left\{ \begin{array}{l} (p^*)_i = \frac{\partial g}{\partial x_i}(x^*) \quad i=1 \dots n-1 \\ F(p^*, z^*, x^*) = 0. \end{array} \right. \\ & \textcircled{N} \end{aligned}$$

\textcircled{P} and \textcircled{N} are called compatibility conditions.

Any triple $(p^*, z^*, x^*) \in \mathbb{R}^{2n+1}$ that

satisfies \textcircled{P} and \textcircled{N} is called admissible,

Note that z^* is uniquely determined by x^* and the boundary data g . But p^* satisfying \textcircled{N} might not \exists . Or, it might \exists but not be unique.

Assume we have (p^*, z^*, x^*) admissible. We want to find a neighborhood of x^* in \mathbb{R} so that we have an admissible triple for each $y \in \text{Nbd}$.

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given $y = (y_1, y_2, \dots, y_{n-1}, 0) \in \Gamma$

with y close to x^* we will solve the characteristic ODEs

$$\left\{ \begin{array}{l} \frac{d\vec{p}}{ds} = -D_x F(\vec{p}(s), z(s), \vec{x}(s)) - D_z F(\vec{p}(s), z(s), \vec{x}(s)) \vec{p}(s) \\ \frac{dz}{ds} = D_p F(\vec{p}(s), z(s), \vec{x}(s)) \cdot \vec{p}(s) \\ \frac{d\vec{x}}{ds} = D_p F(\vec{p}(s), z(s), \vec{x}(s)) \end{array} \right.$$

with initial data

$$\vec{p}(0) = q(y) \quad z(0) = g(y) \quad \vec{x}(0) = y$$

but what is $q(y)$?

We seek $q: \Gamma \rightarrow \mathbb{R}^n$ such that

$$q(x^*) = p^*$$

and $(\vec{q}(y), g(y), y)$ is admissible for $y \in \Gamma$. That is, the compatibility conditions

$$\left\{ \begin{array}{l} (q(y))_i = \frac{\partial \varphi}{\partial x_i}(y) \quad i=1, \dots, n-1 \\ F(\vec{q}(y), g(y), y) = 0 \end{array} \right.$$

hold. If we can find such \vec{q} then we'll be able to solve the characteristic ODEs for initial data with y in a neighborhood of $x^* \in \Gamma$.

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Lemma (Noncharacteristic boundary conditions)

Assume $\frac{\partial F}{\partial p_n}(p^*, z^*, x^*) \neq 0$.

Then $\exists!$ solution \vec{q}_r of

$$\begin{cases} \vec{q}_r(x^*) = p^* \\ (\vec{q}_r(y))_i = \frac{\partial g}{\partial x_i}(y) \quad i=1 \dots n-1 \\ F(\vec{q}_r(y), g(y), y) = 0 \end{cases}$$

for all $y \in \Gamma$ sufficiently close to x^* .

the admissible triple (p^*, z^*, x^*) is called
noncharacteristic if $\frac{\partial F}{\partial p_n}(p^*, z^*, x^*) \neq 0$.

The proof will require the implicit function theorem,
which is in your book (page 634).

proof of lemma:

Given y close to x^* we seek $\vec{q}_r(y)$ such that
the n equations hold:

$$\begin{cases} 0 = (\vec{q}_r(y))_i - \frac{\partial g}{\partial x_i}(y) \\ 0 = F(\vec{q}_r(y), g(y), y) \end{cases}$$

where F , g , and $\frac{\partial g}{\partial x_i}$ are all given. The implicit function theorem is good for telling us when a level set can be viewed as a graph, so..

We introduce $G: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$
defined by

$$G(\vec{p}, \vec{y}) : (G^1(\vec{p}, \vec{y}), G^2(\vec{p}, \vec{y}), \dots, G^n(\vec{p}, \vec{y}))$$

where

$$\begin{cases} G^i(\vec{p}, \vec{y}) = p_i - \frac{\partial E}{\partial x_i}(y) & i=1 \dots n-1 \\ G^n(\vec{p}, \vec{y}) = F(\vec{p}, g(\vec{y}), \vec{y}) \end{cases}$$

note that if $(\vec{p}, g(\vec{y}), \vec{y})$ is admissible then

$$G(\vec{p}, \vec{y}) = \vec{0} \quad \text{so we're seeking}$$

(\vec{p}, \vec{y}) such that $G(\vec{p}, \vec{y}) = \vec{0}$ and we want to
write \vec{p} as a function of \vec{y} .

Fact 1: $G(p^*, x^*) = \vec{0}$ by construction

Fact 2:

$$D_p G(p^*, x^*) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & 0 & \ddots & 0 \\ \frac{\partial E}{\partial p_1}(p^*, x^*) & \frac{\partial E}{\partial p_2}(p^*, x^*) & \frac{\partial E}{\partial p_n}(p^*, x^*) \end{pmatrix}$$

and $\det(D_p G(p^*, x^*)) = \frac{\partial E}{\partial p_n}(p^*, x^*) \neq 0$ by
assumption. We have by the implicit function
theorem: we can uniquely solve $G(p, y) = 0$
for $p = q(y)$ for y in a nbhd of x^* .

Okay, now we're ready to go. We're going from admissible initial data at one point (p^*, z^*, x^*) in $\mathbb{R}^n \times \mathbb{R} \times \Gamma$ to a "patch" of admissible initial data in $\mathbb{R}^n \times \mathbb{R} \times \Gamma$ $(p(y), z(y), y)$ where y is a neighborhood of x^* .

Idea: for each triple $(p(y), z(y), y)$ solve the characteristic ODEs. This will result in

$$z(y; s)$$

being defined for $s \in (-\varepsilon, \varepsilon)$.

Note that $z(y; s)$ has n independent variables, $n-1$ from y , and 1 from s so there's hope to define $u(\vec{x})$ from $z(y; s)$ as long as \vec{x} is reached by a characteristic $\vec{x}(y; s)$ for some y in the neighborhood.

Notation: Let

$$\vec{p}(y; s), z(y; s), \vec{x}(y; s)$$

be the solution of the characteristic ODEs with initial data $(q(y), g(y), y)$.

First we prove that the characteristics cover a neighborhood of x^* in Γ .

Lemma (local invertibility) Assume the non characteristic condition holds:

$$\frac{\partial F}{\partial p_n}(p^*, z^*, x^*) \neq 0$$

Then \exists open interval $I \subset \mathbb{R}$ containing 0 , \exists nbd W of $x^* \in \Gamma \subseteq \mathbb{R}^{n-1}$, and a neighbourhood V of x^* in \mathbb{R}^n such that for each $x \in V$

\exists ! $s \in I$, $y \in W$ such that

$$x = \vec{x}(y; s).$$

Furthermore, the mapping $x \rightarrow (s, y)$ is C^2 .

This is the lemma that tells us that (at least locally) the characteristics are covering a neighbourhood of x^* in a 1:1 and onto manner.

proof of lemma: We want to write

s and y as a function of x when x is in a nbd of x^* :

$s(x), y(x)$ such that

$$\vec{x}(y(x); s(x)) = x$$

That is, we want to invert the mapping

$$\vec{x} : -W \times I \rightarrow V \text{ where } (y, s) \mapsto x.$$

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The inverse function theorem (page 632) guarantees that the mapping is invertible in a neighbourhood of \vec{x}^* .

If $\det(D\vec{x}) \neq 0$ at $(\vec{x}^*, 0)$. Further, the inverse is as smooth as the mapping itself. In our case, the mapping is C^2 so the inverse is also C^2 . (Why is $(\vec{y}, s) \rightarrow \vec{x} \in C^2$? You need to think about the smoothness of solutions of ODEs as well as their smooth dependence on initial data).

So all we need is the Jacobian of

$$\vec{y}, s \rightarrow \vec{x}$$

at $\vec{y} = \vec{x}^*$, $s = 0$. i.e. $\frac{\partial}{\partial y_i} \vec{x}(\vec{y}; s)$ and $\frac{\partial}{\partial s} \vec{x}(\vec{y}; s)$.

First, the y derivatives. Fix $s = 0$.

$$\vec{x}(\vec{y}; 0) = \vec{y} \Rightarrow \left. \frac{\partial (\vec{x})_i}{\partial y_j} \right|_{(\vec{y}^*, 0)} = \delta_{ij} \quad \text{if } i=1 \dots n$$

If $i=n$ then recall that $(\vec{x}(\vec{y}; 0))_n = 0$ because $\Gamma = \{x_n=0\}$. So we have the first $n-1$ columns

of $D\vec{x}(\vec{x}^*, 0)$: the i th column $= \frac{\partial \vec{x}}{\partial y_i} = e^i$

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Now we need the n^{th} column of $D\vec{x}(x^*, 0)$

$$\left. \frac{\partial}{\partial s} \left(\vec{x}(y, s) \right)_i \right|_{(x^*, 0)} = \left. \frac{\partial F}{\partial p_i} (\vec{p}(y; s), \vec{z}(y; s), \vec{x}(y; s)) \right|_{(x^*, 0)}$$

$$= \left. \frac{\partial F}{\partial p_i} (p^*, z^*, x^*) \right.,$$

$$D\vec{x}(y, s) \Big|_{(x^*, 0)} = \begin{pmatrix} 1 & 0 & 0 & \frac{\partial F}{\partial p_1} (p^*, z^*, x^*) \\ 0 & 1 & 0 & \frac{\partial F}{\partial p_2} (p^*, z^*, x^*) \\ 0 & 0 & \ddots & \vdots \\ \vdots & & & \vdots \\ 0 & 0 & 0 & \frac{\partial F}{\partial p_n} (p^*, z^*, x^*) \end{pmatrix}$$

and $\det(D\vec{x}(x^*, 0)) = \frac{\partial F}{\partial p_n} (p^*, z^*, x^*) \neq 0$
by the noncharacteristic condition.

This proves that

$$y, s \Rightarrow x$$

given by $x = \vec{x}(y, s)$ is invertible, w/ C^2 inverse,
in a neighbourhood of x^* . //

Now we're done! We've shown that
in a neighbourhood V of x^* in \mathbb{R}^n
we can uniquely solve

$$x = \vec{x}(y, s)$$

resulting in $y = \vec{y}(x)$ and $s = s(x)$. This
is precisely what we did in our examples;
given $x \in \Gamma$ we found $y \in \Gamma$ and s such that
 $x = \vec{x}(y, s)$. We then used this to write
 $\vec{z}(y; s)$ as $u(x)$.

Theorem: (Local existence theorem). If $x \in V$
and $y(x)$ and $s(x)$ are the inverses
of the mapping $y, s \rightarrow x$ from the
previous lemma, let

$$u(x) := \vec{z}(y(x); s(x))$$

Then u is in $C^2(V)$ and solves the
PDE $F(Du, u, x) = 0$ in V and satisfies
the boundary condition $u = g$ on $\Gamma \cap V$.

Proof:

1. Fix $y \in W$ the neighbourhood of x^0 in \mathbb{R} . Solve the characteristic ODEs resulting in $\vec{p}(y; s)$, $z(y; s)$, and $\vec{x}(y; s)$.

2. We now show that

$$\begin{aligned} f(y, s) &:= F(\vec{p}(y; s), z(y; s), \vec{x}(y; s)) \\ &= 0 \quad \text{for all } s \in I. \end{aligned}$$

$$\begin{aligned} \text{Note that } f(y, 0) &= F(\vec{p}(y; 0), z(y; 0), \vec{x}(y; 0)) \\ &= F(q^*(y), g^*(y), y) \\ &= 0 \quad \text{by the compatibility condition.} \end{aligned}$$

Now, we argue that $\frac{\partial f}{\partial s} = 0$ on I . This, combined w/ the continuity of f will give that $f(y; s) = 0$ for all $s \in I$

$$\begin{aligned} \frac{\partial}{\partial s} f(y, s) &= \left(\sum_{i=1}^n \frac{\partial F}{\partial p_i} \frac{dp_i}{ds} \right) + \frac{\partial F}{\partial z} \frac{dz}{ds} + \sum_{i=1}^n \frac{\partial F}{\partial x_i} \frac{dx_i}{ds} \\ &= \left(\sum_{i=1}^n \frac{\partial F}{\partial p_i} \left(-\frac{\partial F}{\partial x_i} - p_i \frac{\partial F}{\partial z} \right) \right) + \frac{\partial F}{\partial z} \sum_{i=1}^n \frac{\partial F}{\partial p_i} p_i \\ &\quad + \sum_{i=1}^n \frac{\partial F}{\partial x_i} \frac{\partial F}{\partial p_i} \quad (\text{by characteristic ODEs}) \end{aligned}$$

and so $\frac{\partial}{\partial s} f(y, s) = 0$ as claimed.

3. By construction, we have

$$F(p(x), u(x), x) = 0 \quad \text{for all } x \in V$$

where $p(x) := \vec{p}(y(x), s(x))$

$$u(x) := z(y(x), s(x))$$

and $y(x)$ and $s(x)$ come from the inverse mapping of the previous lemma. If we know that $p(x) = Du(x)$ for all $x \in V$ then we'd be done!

4. In this direction, we want to understand what $\frac{\partial z}{\partial y_i}$ and $\frac{\partial z}{\partial s}$ are. Since they're going to show up if you try to show

$$Du(x) = D_x z(y(x), s(x))$$

$$\frac{\partial}{\partial s} z(y, s) = D_p F(p(y, s), z(y, s), x(y, s)) \cdot p(y, s)$$

by characteristic ODE for $\frac{dz}{ds}$

$$= \left(\frac{\partial}{\partial s} x(y, s) \right) \cdot p(y, s)$$

by characteristic ODE for $\frac{d\vec{x}}{ds}$

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I claim that for each $i=1\dots n-1$

$$\frac{\partial}{\partial y_i} z(y; s) = \sum_{j=1}^n (p(y; s))_j \frac{\partial (x(y; s))_j}{\partial y_i}$$

for all $y \in W$, $s \in I$

Fix y and let

$$r_i(s) := \frac{\partial}{\partial y_i} z(y; s) - \sum_{j=1}^n (p(y; s))_j \frac{\partial}{\partial y_i} (x(y; s))_j.$$

It suffices to show that $r_i(s) = 0$ for all $s \in I$.

This will follow by

(1) $r_i(0) = 0$, (2) $\frac{d}{ds} r_i(s) = 0$, (3) continuity of r_i on I .

$$(1) \quad r_i(0) = \left. \frac{\partial}{\partial y_i} z(y; s) \right|_{(y; 0)} - \sum_{j=1}^n (p(y; 0))_j \left. \frac{\partial}{\partial y_i} (x(y; s))_j \right|_{(y; 0)}$$

$$\text{know } \left. \frac{\partial z}{\partial y_i} \right|_{(y, 0)} = \left. \frac{\partial g}{\partial x_i} \right|_{(y, 0)}$$

$$\text{and } \left. \frac{\partial}{\partial y_i} (x(y; s))_j \right|_{(y, 0)} = \delta_{ij}$$

$$\Rightarrow r_i(0) = \left. \frac{\partial g}{\partial x_i} (y) - (p(y; 0))_i \right|_i = \left. \frac{\partial g}{\partial x_i} (y) - (q(y))_i \right|_i$$

$= 0$ by compatibility
homomorphism

this holds for all $y \in W$, $s \in I$

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$$\textcircled{2} \quad \frac{d}{ds} r_i(s) = \frac{\partial^2 z}{\partial y_i \partial s}(y; s) - \sum_{j=1}^n \left[\frac{\partial p^j}{\partial s}(y; s) \frac{\partial x^j}{\partial y_i} + p^j \frac{\partial^2 x^j}{\partial y_i \partial s} \right]$$

We've already shown that

$$\frac{\partial z}{\partial s}(y; s) = \sum_{j=1}^n p^j(y; s) \frac{\partial x^j}{\partial s}(y; s) \quad \text{for } y \in W, s \in I$$

so let's differentiate that wrt y_i :

$$\frac{\partial^2 z}{\partial y_i \partial s} = \sum_{j=1}^n \left[\frac{\partial p^j}{\partial y_i} \frac{\partial x^j}{\partial s} + p^j \frac{\partial^2 x^j}{\partial y_i \partial s} \right]$$

$$\text{so } \frac{d}{ds} r^i(s) = \sum_{j=1}^n \left[\frac{\partial p^j}{\partial y_i} \frac{\partial x^j}{\partial s} - \frac{\partial p^j}{\partial s} \frac{\partial x^j}{\partial y_i} \right]$$

now use the characteristic ODE

for $\frac{dx}{ds}$ and $\frac{dp}{ds}$

$$= \sum_{j=1}^n \left[\frac{\partial p^j}{\partial y_i} \frac{\partial F}{\partial p_j} - \frac{\partial x^j}{\partial y_i} \left(-\frac{\partial F}{\partial x_j} - \frac{\partial F}{\partial z} p_j \right) \right]$$

$$\text{so } \frac{d}{ds} r^i = \sum_{j=1}^n \left[\frac{\partial p^j}{\partial y_i} \frac{\partial F}{\partial p_j} + \frac{\partial x^j}{\partial y_i} \frac{\partial F}{\partial x_j} + \frac{\partial x^j}{\partial y_i} \frac{\partial F}{\partial z} p_j \right]$$

taking a deep breath, we realize that the above looks like $\frac{\partial}{\partial y_i} F(p(y; s), z(y; s), x(y; s))$

so let's see if we can use that...

we know that

$$F(p(y; s), z(y; s), x(y; s)) = 0 \quad \text{for } y \in W, s \in I$$

$$\Rightarrow \sum_{j=1}^n \frac{\partial F}{\partial p_j} \frac{\partial p_j}{\partial y_i} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y_i} + \sum_{j=1}^n \frac{\partial F}{\partial x_j} \frac{\partial x_j}{\partial y_i} = 0$$

Using this to replace

$$\sum_{j=1}^n \frac{\partial F}{\partial p_j} \frac{\partial p_j}{\partial y_i} \quad \text{above,}$$

$$\frac{d}{ds} r^i(s) = \frac{\partial F}{\partial z} \left[\sum_{j=1}^n p^j \frac{\partial x^j}{\partial s} - \frac{\partial z}{\partial y_i} \right] = - \frac{\partial F}{\partial z} r^i(s)$$

$$\Rightarrow r^i(s) = r^i(0) e^{- \int_0^s \frac{\partial F}{\partial z}(p(y; \tau), z(y; \tau), x(y; \tau)) d\tau} \\ = 0 \quad \text{since } r^i(0) = 0$$

this shows that

$$\frac{\partial z}{\partial y_i}(y; s) = \sum_{j=1}^n p^j(y; s) \frac{\partial x^j}{\partial y_i}(y; s) \quad \text{for } i = 1 \dots n-1$$

as desired. This identity holds for $s \in I, y \in W$.

5. we now prove that

$$\vec{p}(y(x), s(x)) = p(x) = Du(x) \quad \text{for all } x \in V.$$

If we can do this, we're done.

$$\text{recall } u(x) = z(y(x), s(x))$$

$$\frac{\partial u}{\partial x_i} = \frac{\partial z}{\partial s} \frac{\partial s}{\partial x_i} + \sum_{j=1}^{n-1} \frac{\partial z}{\partial y_j} \frac{\partial y_j}{\partial x_i}$$

$$= \left[\sum_{k=1}^n p^k \frac{\partial x^k}{\partial s} \right] \frac{\partial s}{\partial x_i} + \sum_{j=1}^{n-1} \left[\sum_{k=1}^n p^k \frac{\partial x^k}{\partial y_j} \right] \frac{\partial y_j}{\partial x_i}$$

by the two identities we proved
for $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial y_j}$

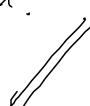
$$= \sum_{k=1}^n p^k \left[\frac{\partial x^k}{\partial s} \frac{\partial s}{\partial x_i} + \sum_{j=1}^{n-1} \frac{\partial x^k}{\partial y_j} \frac{\partial y_j}{\partial x_i} \right]$$

$$= \sum_{k=1}^n p^k \frac{\partial}{\partial x_i} x^k \quad \text{since } x^k(y(x), s(x))$$

$$= \sum_{k=1}^n p^k \delta_{ik} = p^i(x) \quad \text{for all } x \in V.$$

This shows that $p(x) := \vec{p}(y(x), s(x))$

satisfies $p(x) = Du(x)$ as desired.



Note: go back to the example $u_{x_1} u_{x_2} = u$ in $\Gamma =$

$$u(0, x_2) = x_2 \text{ on } \Gamma$$

$$\text{then } F(p_1, p_2, z, x_1, x_2) = p_1 p_2 - z$$

$\frac{\partial F}{\partial p_1} = p_2 \neq 0$ if $x_2 \neq 0$ noncharacteristic condition fails at $(0, 0)$!