

Characteristics...

Theorem: If  $u \in C^2(\bar{U})$  solves the PDE  $F(Du, u, x) = 0$  in  $\bar{U}$  and  $\vec{x}(s)$  satisfies the ODE

$$\frac{d\vec{x}}{ds} = D_p F(\vec{p}(s), z(s), \vec{x}(s))$$

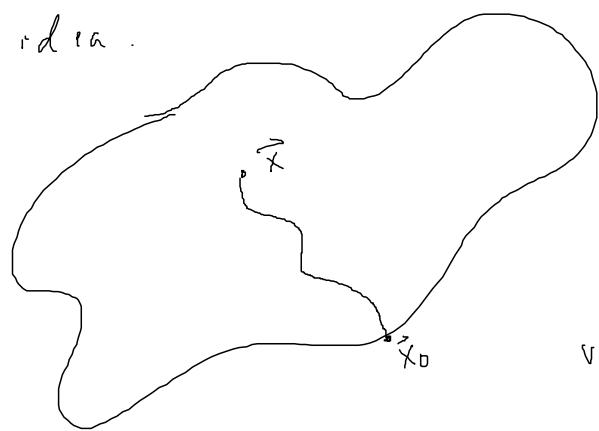
where  $\vec{p}(s) = Du(\vec{x}(s))$  and  $z(s) = u(\vec{x}(s))$  then  $\vec{p}(s)$  and  $z(s)$  satisfy the ODEs

$$\frac{d\vec{p}}{ds} = -D_x F(\vec{p}(s), z(s), \vec{x}(s)) - \vec{p}(s) D_z F(\vec{p}(s), z(s), \vec{x}(s))$$

$$\frac{dz}{ds} = D_p F(\vec{p}(s), z(s), \vec{x}(s)) \cdot \vec{p}(s)$$

Note: this is saying that if you have a  $C^2$  solution  $u$  and a trajectory  $\vec{x}(s)$  then  $\vec{p}(s)$  and  $z(s)$  will have to satisfy the other 2 ODEs. This is different from what we're trying to do which is to find  $u(\vec{x})$  by solving the 3 ODEs  $\frac{d\vec{x}}{ds}, \dots, \frac{d\vec{p}}{ds}, \dots, \frac{dz}{ds}, \dots$

Recall the idea.



$\vec{x} \in \bar{U}$  fixed.

$\vec{x}(s)$  is the characteristic through  $\vec{x}$ , and  $\vec{x}(s)$  hits  $\Gamma$  at  $\vec{x}_0$ .

use  $g(\vec{x}_0)$  and  $\frac{dz}{ds} = \dots$  to find  $u(\vec{x})$

(2)

$$\text{Ex: } \left\{ \begin{array}{l} x_1 \frac{\partial u}{\partial x_2} - x_2 \frac{\partial u}{\partial x_1} = u \quad \text{in } \mathcal{D} \\ u = f \quad \text{on } \Gamma \end{array} \right.$$

$$\text{where } \mathcal{D} = \{x_1 > 0, x_2 > 0\} \quad \Gamma = \{x_2 = 0, x_1 > 0\}$$

$$F(\vec{p}, z, \vec{x}) = x_1 p_2 - x_2 p_1 - z$$

$$D_p F = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad D_z F = -1 \quad D_x F = \begin{pmatrix} p_2 \\ -p_1 \end{pmatrix}$$

$$\text{so } \left\{ \begin{array}{l} \frac{d\vec{p}}{ds} = \begin{pmatrix} p_1(s) - p_2(s) \\ p_1(s) + p_2(s) \end{pmatrix} \\ \frac{dz}{ds} = x_1(s)p_2(s) - x_2(s)p_1(s) \\ \frac{d\vec{x}}{ds} = \begin{pmatrix} -x_2(s) \\ x_1(s) \end{pmatrix} \end{array} \right.$$

the ODE for  $\vec{x}(s)$  can be solved immediately.

the ODE for  $z(s)$  looks like it's coupled to  $\vec{p}(s)$ . i.e.

you can't solve it w/o knowing  $\vec{p}(s)$  somehow.

But remember:  $x_1(s)p_2(s) - x_2(s)p_1(s) = z(s)$  from the PDE!

$$\frac{dz}{ds} = z(s) \quad \text{is solvable.}$$

$\Rightarrow$  Don't need to bother w/ ODE for  $\vec{p}(s)$ .

(3)

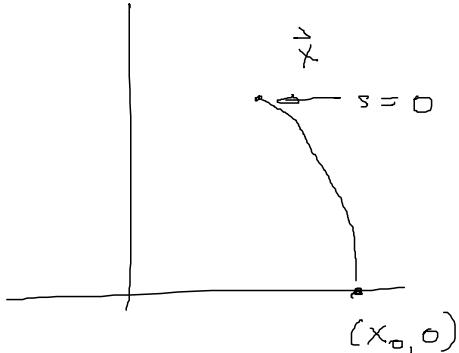
solve

$$\begin{cases} \frac{d\vec{x}}{ds} = \begin{pmatrix} -x_2(s) \\ x_1(s) \end{pmatrix} \\ \frac{dz}{ds} = z(s) \end{cases}$$

with initial data

$$\vec{x}(0) = \vec{x}$$

$$z(0) = u(\vec{x})$$

Abusing notation,  $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ 

and

$$\begin{cases} \vec{x}(s) = \begin{pmatrix} x_1 \cos(s) - x_2 \sin(s) \\ x_1 \sin(s) + x_2 \cos(s) \end{pmatrix} \\ z(s) = u(\vec{x}) e^s \end{cases}$$

such  $\tilde{s}$  such that  $\vec{x}(\tilde{s}) \in \Gamma$ .

i.e.  $x_1 \sin(\tilde{s}) + x_2 \cos(\tilde{s}) = 0 \Rightarrow \tilde{s} = \arctan\left(-\frac{x_2}{x_1}\right)$

at this value of  $\tilde{s}$ ,

$$\vec{x}(\tilde{s}) = \begin{pmatrix} \sqrt{x_1^2 + x_2^2} \\ 0 \end{pmatrix}$$

$$\Rightarrow z(\tilde{s}) = u(\sqrt{x_1^2 + x_2^2}, 0) = g(\sqrt{x_1^2 + x_2^2})$$

by definition of  $z(s)$

but  $z(\tilde{s}) = u(\vec{x}) e^{\arctan\left(-\frac{x_2}{x_1}\right)}$

$$\Rightarrow \text{solving for } u(\vec{x}) \Rightarrow u(\vec{x}) = g(\sqrt{x_1^2 + x_2^2}) e^{-\arctan\left(-\frac{x_2}{x_1}\right)}$$

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(4)

Now, you can check that

$$u(\vec{x}) = \tilde{g}(\sqrt{x_1^2 + x_2^2}) e^{\arctan\left(\frac{x_2}{x_1}\right)}$$

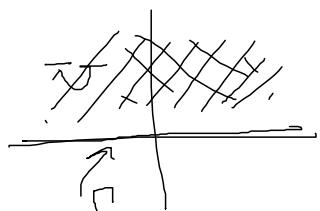
actually solves the PDE.

Here, we solved the PDE by taking  $x \in \bar{U}$  and tracing the characteristic all the way back to  $\Gamma$ . But what if this had failed? What if the characteristic stopped existing before it reached  $\partial U$ ? Or what if it hit  $\partial U$  but not in  $\Gamma$ ?

(recall: the characteristics are solutions to ODEs. You're not guaranteed solutions that exist for all  $s$ ... they might only exist for an interval of  $s$  values.)

In general, we approach these problems differently. That is, we start at  $\Gamma$  and flow into  $U$ . Rather than trying to get from  $U$  to  $\Gamma$ . Here's an example.

$$\begin{cases} \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = u^2 & \text{in } U \\ u = g & \text{on } \Gamma \end{cases}$$



$$U = \{x_2 > 0\} \subseteq \mathbb{R}^2, \quad \Gamma = \{x_2 = 0\} \subseteq \mathbb{R}^2$$

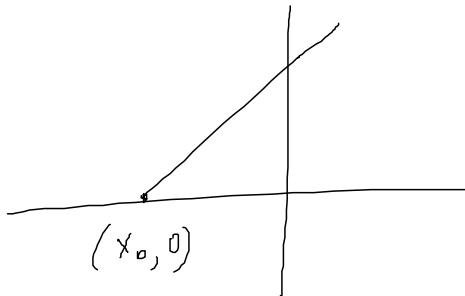
$$F(\vec{p}, \vec{z}, \vec{x}) = p_1 + p_2 - z^2$$

$$D_p F = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad D_z F = -2z \quad D_x F = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{array}{l} \frac{d\vec{p}}{ds} = \begin{pmatrix} 2z(s) \\ 2z(s) \end{pmatrix} \\ \frac{d\vec{z}}{ds} = p_1(s) + p_2(s) = \vec{z}(s)^2 \\ \frac{d\vec{x}}{ds} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array} \right.$$

by PDE

Again, we can solve for  $\vec{x}(s)$  and  $z(s)$  w/o using  $\vec{p}(s)$ .



$$\left\{ \begin{array}{l} \vec{x}(s) = \begin{pmatrix} x_0 + s \\ s \end{pmatrix} \\ z(s) = \frac{z(0)}{1 - s z(0)} \end{array} \right.$$

$$\text{know } z(0) = g(x_0) \Rightarrow z(s) = \frac{g(x_0)}{1 - s g(x_0)}$$

$$\text{given } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in T \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{x}(s) \quad \text{for } \tilde{s} = x_2 \Rightarrow x_2 = x_1 - x_2$$

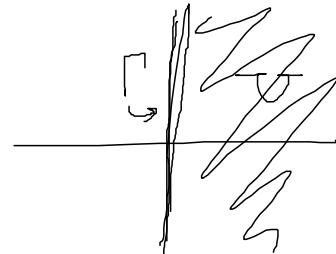
$$\Rightarrow z(\tilde{s}) = \boxed{\frac{g(x_1 - x_2)}{1 - x_2 g(x_1 - x_2)} = u(x_1, x_2)}$$

Now, check that  $u(x_1, x_2)$  solves the PDE in  $T$ .

6

Here's an example where the PDE is fully nonlinear

$$\begin{cases} u_{x_1} u_{x_2} = u & \text{in } \mathcal{V} \\ u(x_1, x_2) = x_2^2 & \text{in } \mathcal{P} \end{cases}$$



$$\text{where } \mathcal{V} = \{x_1 \geq 0\} \subseteq \mathbb{R}^2 \quad \mathcal{P} = \{x_1 = 0\}$$

$$F(\vec{p}, z, \vec{x}) = p_1 p_2 - z$$

$$D_p F = \begin{pmatrix} p_2 \\ p_1 \end{pmatrix} \quad D_z F = -1 \quad D_x F = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \frac{d\vec{p}}{ds} = \vec{p}(s) \\ \frac{dz}{ds} = 2p_1(s)p_2(s) \\ \frac{d\vec{x}}{ds} = \begin{pmatrix} p_2(s) \\ p_1(s) \end{pmatrix} \end{cases}$$

To find  $\vec{x}(s)$ , we need to solve for  $\vec{p}(s)$ . Fortunately we can do that.

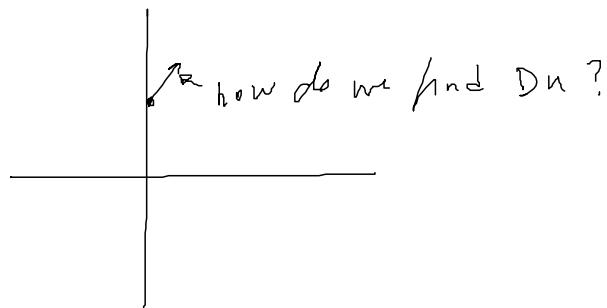
$$\begin{cases} p_1(s) = p_1(0)e^s \\ p_2(s) = p_2(0)e^s \end{cases}$$

but what is the initial data for  $\vec{p}$ ?

7

Recall,  $\vec{p}(s) = Du(\vec{x}(s))$

$$\vec{p}(0) \rightarrow Du \text{ on } \Gamma$$



Observation 1: we know  $u(0, x_0) = x_0^2$

so we can at least figure out the derivatives in the  $\Gamma$  direction using the specified data.

$$\frac{\partial u}{\partial x_2}(0, x_0) = 2x_0 \Rightarrow \boxed{p_2(0) = 2x_0}$$

To find  $p_1(0)$  we use the PDE

$$p_1(0)p_2(0) = u(0, x_0) = x_0^2$$

$$\Rightarrow \boxed{p_1(0) = \frac{x_0}{2}}$$

So  $\vec{p}(s) = \begin{pmatrix} \frac{x_0}{2} e^s \\ 2x_0 e^s \end{pmatrix}$

Note: if  $\Gamma$  were codimension 1 in  $\mathbb{R}^n$ , you'd hope to use the initial data on  $\Gamma$  to find  $n-1$  components of  $\vec{p}(0)$  and use the PDE to determine the remaining comp.

8

$$\frac{d\vec{x}}{ds} = \begin{pmatrix} p_2(s) \\ p_1(s) \end{pmatrix} = \begin{pmatrix} 2x_0 e^s \\ \frac{x_0}{2} e^s \end{pmatrix}$$

$$\Rightarrow \vec{x}(s) = \begin{pmatrix} 2x_0(e^s - 1) \\ \frac{x_0}{2}(e^s - 1) + x_0 \end{pmatrix}$$

note:  $\vec{x}(0) = \begin{pmatrix} 0 \\ x_0 \end{pmatrix}$

$$\therefore \vec{x}(s) = \begin{pmatrix} 2x_0(e^s - 1) \\ \frac{x_0}{2}(e^s + 1) \end{pmatrix}$$

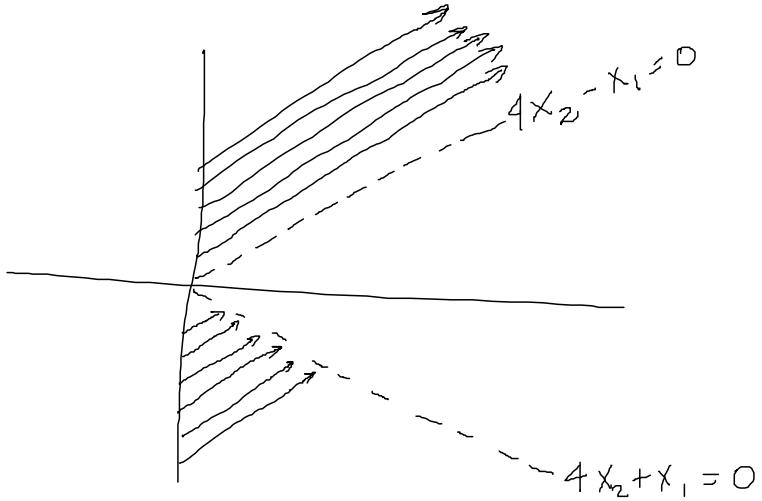
$$\frac{dz}{ds} = 2p_1(s)p_2(s) = 2x_0^2 e^{2s}$$

$$\Rightarrow z(s) = x_0^2 e^{2s}$$

Now, given  $(x_1, x_2) \in U$ , seek  $x_0$  and  $\tilde{s}$   
 so that  $\vec{x}(\tilde{s}) = (x_1, x_2)$

$$\Rightarrow x_0 = \frac{4x_2 - x_1}{4} \quad e^{\tilde{s}} = \frac{4x_2 + x_1}{4x_2 - x_1}$$

does this make sense? First let's look at the characteristics



assume  $x_0 > 0$  then as  $s$  increases from 0,  $\vec{x}(s)$

enters  $T$ , and  $4x_2(s) - x_1(s) = x_0$

so for  $x_0 > 0$  the characteristics are rays tending to  $(\infty, \infty)$

if  $x_0 < 0$  then as  $s$  decreases from 0,  $\vec{x}(s)$  enters  $T$ .

and  $4x_2(s) - x_1(s) = 4x_0$ . But. as  $s \rightarrow -\infty$ ,

$$\vec{x}(s) \rightarrow \begin{pmatrix} -2x_0 \\ \frac{x_0}{2} \end{pmatrix} \text{ which is on the line } 4x_2 + x_1 = 0$$

so no  $(x_1, x_2)$  in the cone  will be reached by a characteristic.

Anyway, if  $(x_1, x_2)$  is reached by a characteristic then

(10)

$$e^{\tilde{s}} = \frac{4x_2 + x_1}{4x_2 - x_1} \quad \text{and} \quad x_0 = \frac{4x_2 - x_1}{4}$$

$$\Rightarrow z(s) = u(x_1, x_2)$$

||

$$x_0^2 \left( \frac{4x_2 + x_1}{4x_2 - x_1} \right)^2 = \left( \frac{4x_2 - x_1}{4} \right)^2 \left( \frac{4x_2 + x_1}{4x_2 - x_1} \right)^2 = \frac{(4x_2 + x_1)^2}{16}$$

$$u(x_1, x_2) = \frac{(4x_2 + x_1)^2}{16} \quad \text{as } x_1 \rightarrow 0 \quad u(x_1, x_2) \rightarrow x_2^2 \checkmark$$

$$\left( \frac{\partial u}{\partial x_1} \right) + \left( \frac{\partial u}{\partial x_2} \right) = \left( \frac{4x_2 + x_1}{8} \right) + \left( \frac{4x_2 + x_1}{2} \right) = \frac{(4x_2 + x_1)^2}{16}$$

Note we've found a solution in all of  $\mathcal{U}$  even though the characteristics didn't cover all of  $\mathcal{U}$ . Franky.