

We now consider a reaction diffusion equation

$$u_t = u_{xx} + f(u) \quad \text{on } \mathbb{R} \times (0, \infty)$$

where f will be assumed to satisfy the following:

1) $\exists a \in (0, 1)$ such that

$$f(0) = f(a) = f(1)$$

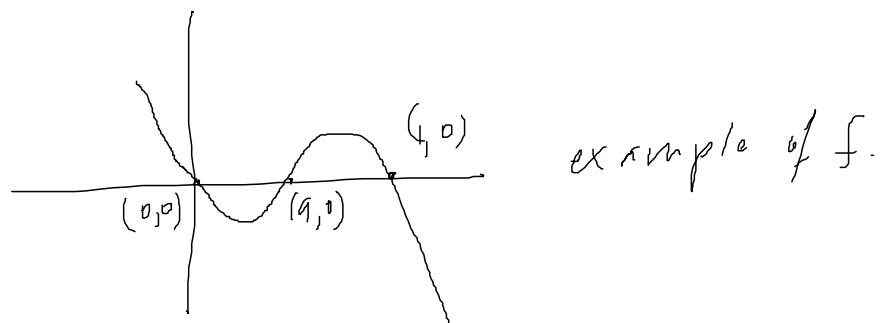
2) $f < 0$ on $(0, a)$, $f > 0$ on $(a, 1)$

3) $f'(0) < 0$, $f'(1) < 0$

4) $\int_0^1 f(z) dz > 0$

5) f is smooth

E.g.

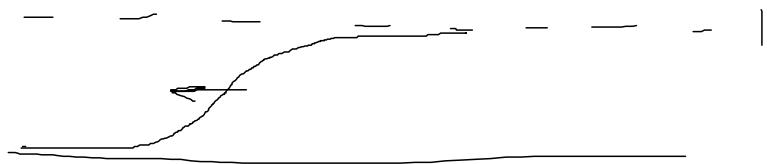


We seek a travelling wave solution

$$u(x, t) = v(x - vt)$$

where $v < 0$ and $u \rightarrow 0$ as $x \rightarrow -\infty$
 $u \rightarrow 1$ as $x \rightarrow \infty$

For example, v = concentration of burnt material



The travelling wave corresponds to a front of combustion moving into the unburnt material.

Unlike the KdV case, which had a continuum of solution solutions, we'll find 3! speed σ .

From the travelling wave ansatz,

$$-\sigma v' = v'' + f(v)$$

$$\text{with } v(\eta) \rightarrow 1 \text{ as } \eta \rightarrow -\infty$$

$$v'(\eta) \rightarrow 0 \text{ as } \eta \rightarrow -\infty$$

$$v(\eta) \rightarrow 1 \text{ as } \eta \rightarrow \infty$$

$$v'(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty$$

— Analyse in the phase plane

$$v'' = -\sigma v' - f(v)$$

$$\Rightarrow \begin{cases} v' = w \\ w' = -\sigma w - f(v) \end{cases}$$

(3)

We have 3 fixed points

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

linearize $\begin{pmatrix} 0 & 1 \\ -f'(v_0) & -\sigma \end{pmatrix}$

stability given by eigenvalues

$$-\frac{\sigma}{2} \pm \frac{\sqrt{\sigma^2 + 4f'(v_0)}}{2}$$

so since $f'(0) < 0$ and $f'(1) < 0$ we know

$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are hyperbolic fixed points.

unstable direction: $\begin{pmatrix} 1 \\ \lambda^+ \end{pmatrix}$ where $\lambda^+ = -\frac{\sigma}{2} + \frac{\sqrt{\sigma^2 + 4f'(v_0)}}{2}$

stable direction: $\begin{pmatrix} 1 \\ \lambda^- \end{pmatrix}$ $\lambda^- = -\frac{\sigma}{2} - \frac{\sqrt{\sigma^2 + 4f'(v_0)}}{2}$

Since $|f'(v_0)| > 0$ the type of the fixed point

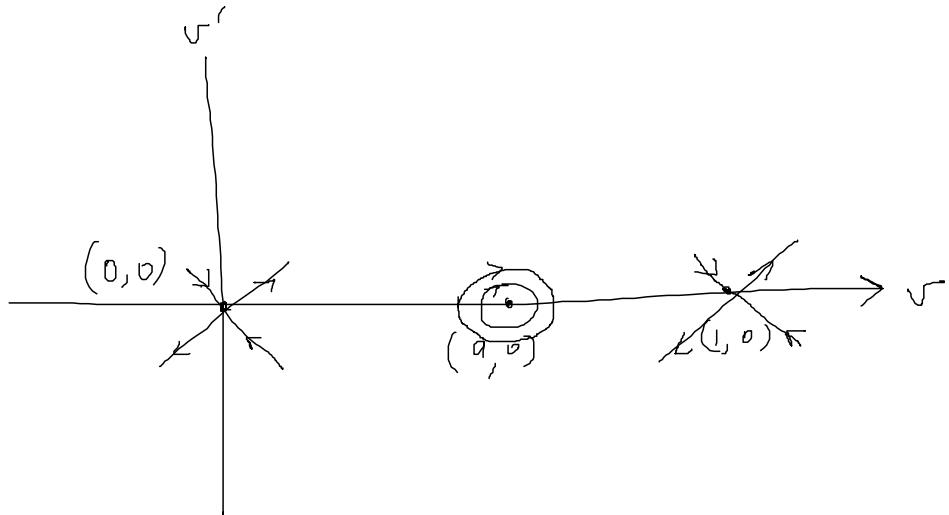
depends on the size of σ . For $\sigma = 0$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

has purely imaginary eigenvalues. \Rightarrow for

$|\sigma|$ small, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has purely imaginary eigenvalues.

(4)

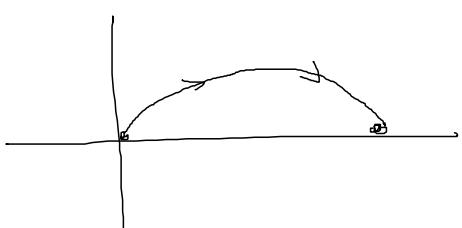
phase plane for $|\sigma| < 1$



$(0,0)$ has an unstable manifold W^u which is tangent to $w = \lambda_0^+ v$ (and has a stable manifold that is tangent to $w = \lambda_0^- v$.)

$(1,0)$ has a stable manifold W^s which is tangent to $w = \lambda_1^- (v-1)$ (and has an unstable manifold that is tangent to $w = \lambda_1^+ (v-1)$.)

We seek a value of σ so that the unstable manifold from $(0,0)$ intersects the stable manifold of $(1,0)$. This heteroclinic orbit provides the profile of a travelling wave solution.



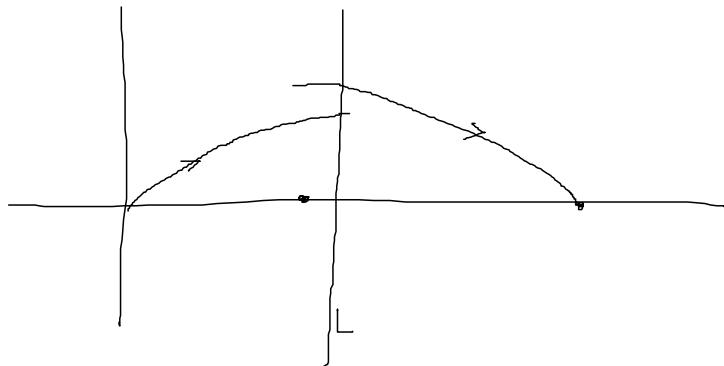
(5)

That is, we want to show that

in $\{v > 0, w > 0\}$ the two manifolds coincide

$$W^u = W^s$$

Start by showing they both intersect the vertical line through $(\alpha + \epsilon, 0)$. (ϵ fixed and small.)



first show that $W^u \cap L \neq \emptyset$, and $W^s \cap L \neq \emptyset$
if $\sigma < 0$

To do this, we seek a Lyapunov function
for the ODE

$$\dot{v} = v'' + \sigma v' + f(v)$$

Multiply by v' and integrate up

$$\mathcal{O} = v'v'' + \sigma v'v' + v'f(v)$$

$$\mathcal{O} = \left(\frac{1}{2}v'\right)^2 + \sigma(v')^2 + \left(\int_0^v f(z) dz\right)'$$

②

$$\Rightarrow a = \frac{1}{2} (v'(\eta))^2 + \sigma \int_{-\infty}^{\eta} [v'(y)]^2 dy + \int_0^{v(\eta)} f(z) dz$$

so we define

$$E(v, w) := \frac{1}{2} w^2 + \int_0^v f(z) dz$$

then

$$\begin{aligned} \frac{d}{dt} E(v(t), w(t)) &= w(t) w'(t) + f(v(t)) w(t) \\ &= w(t) [w'(t) + f(v(t))] \\ &\leq w(t) [-\sigma w(t)] = -\sigma w(t)^2 \end{aligned}$$

and any solution must have the Lyapunov function
 E non decreasing.

So now that we have a Lyapunov function
 $E(v, w)$ we want to draw its level sets.

Obs 1: $E(0, 0) = \frac{1}{2} 0^2 + \int_0^0 f(z) dz = 0$

Obs 2: Since $f < 0$ on $(0, a)$ and $f > 0$ on $(a, 1)$ and
 $\int_0^1 f(z) dz > 0$, $\exists! v_c \in (0, 1)$ s.t. $\int_0^{v_c} f(z) dz = 0$

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Furthermore $v_c > \alpha$.

$\Rightarrow E(v_c, 0) = 0$ and v_c is on the same energy-level as $(0, 0)$

Note: v_c determines the ε that we have in our definition of L . we need $L \perp$ the line of v_c .

Other observations:

if $w > 0$ then the level set through $(0, w)$ is perpendicular to the w -axis.

Similarly, at fixed points $(\alpha, 0)$ and $(1, 0)$:

if $w > 0$ then the level set intersects the line $\begin{pmatrix} 1 \\ z \end{pmatrix}$ perpendicularly and

intersects the line $\begin{pmatrix} 1 \\ z \end{pmatrix}$ perpendicularly.

(Why? if $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$ is tangent to the

level set $E(v, w) = c$ at the point

(v_0, w_0) then $f(v_0)t_1 + w_0t_2 = 0$.)

Similarly, if $(v_0, 0)$ has $v_0 \neq \alpha$ or if f then the level set through $(v_0, 0)$ is \perp to the v axis

Finally, via taylor series expansion near $(0,0)$, if (v,w) is close to $(0,0)$

$$\begin{aligned} \text{then } E(v,w) &\equiv \frac{w^2}{2} + \int_0^v f'(z) + f'(0)z \, dz \\ &= \frac{w^2}{2} + f'(0) \int_0^v z \, dz \\ &= \frac{w^2}{2} + f'(0) \frac{v^2}{2} \end{aligned}$$

if $(v_0, w_0) \approx (0,0)$ and are on the same level set as

$$(0,0) \text{ then } \frac{w_0^2}{2} + f'(0) \frac{v_0^2}{2} = 0$$

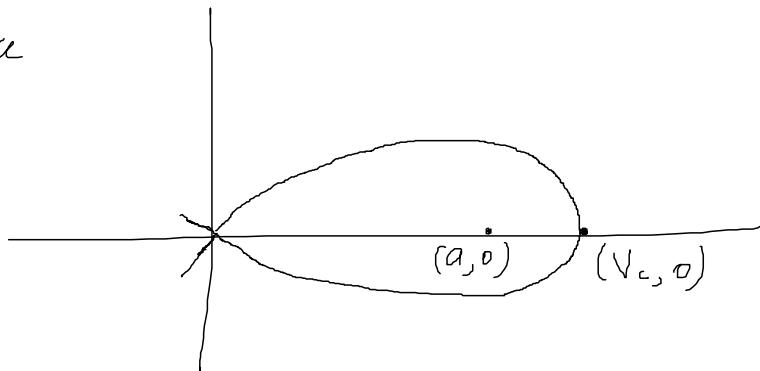
$$\Rightarrow \frac{w_0^2}{2} = -f'(0) \frac{v_0^2}{2}$$

$$\Rightarrow w_0^2 = -f'(0) v_0^2$$

$$\Rightarrow w_0 = \sqrt{-f'(0)} v_0 \quad \text{if } w_0 > 0, v_0 > 0$$

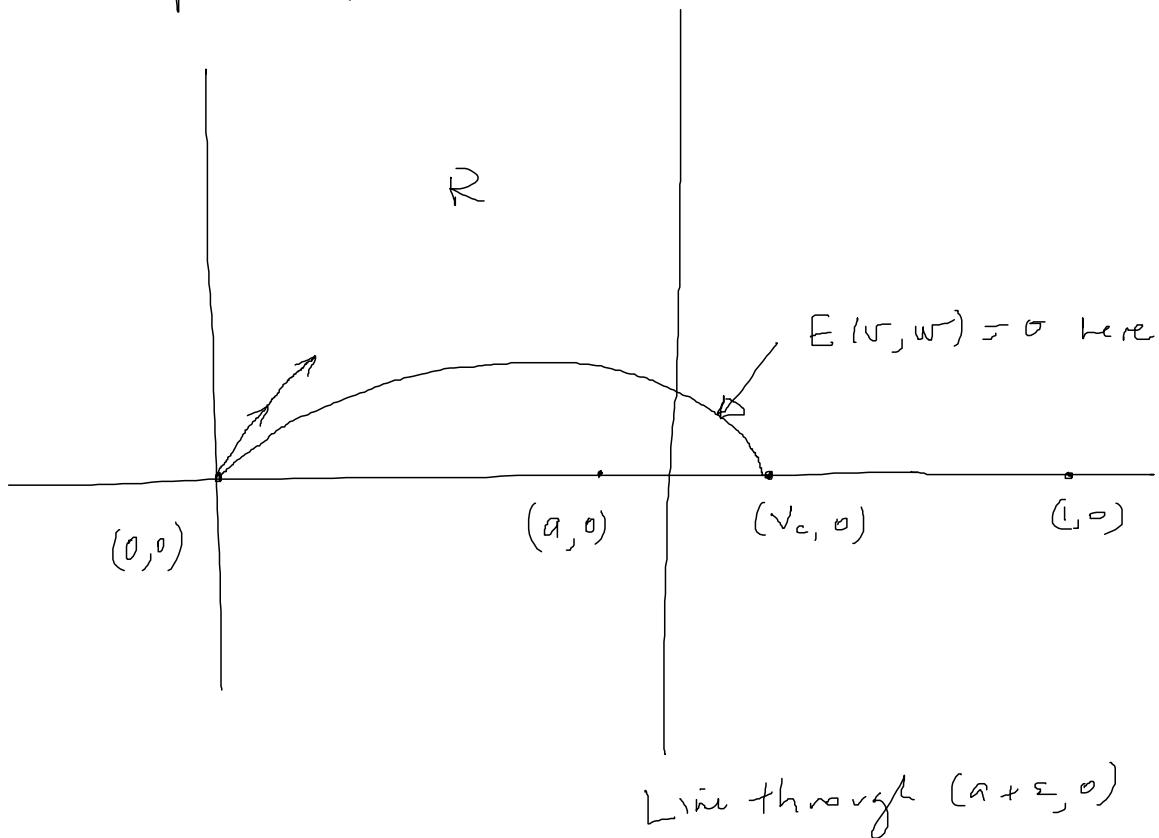
In sum, the level set through $(0,0)$ looks

like



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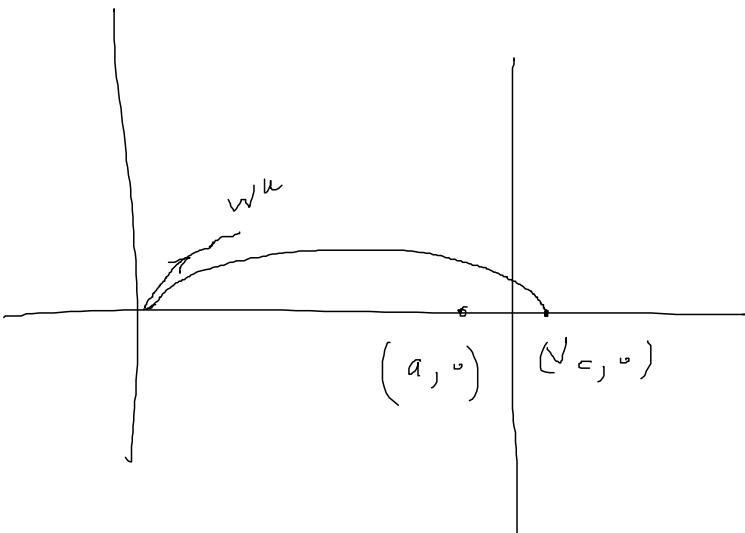
We use this level set to define a trapping region in phase space:



Let R be the open region with vertical left and right boundaries and
 $E(v, w) = 0$ in the bottom.

If $\sigma < 0$,
the unstable manifold cannot cross the bottom boundary. Since it starts out above because
 $\lambda_0^+ > \sqrt{-f'(0)} \Big)$ and can't go below.

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w^u enters R . it can't leave through the left boundary on the left, the vector field

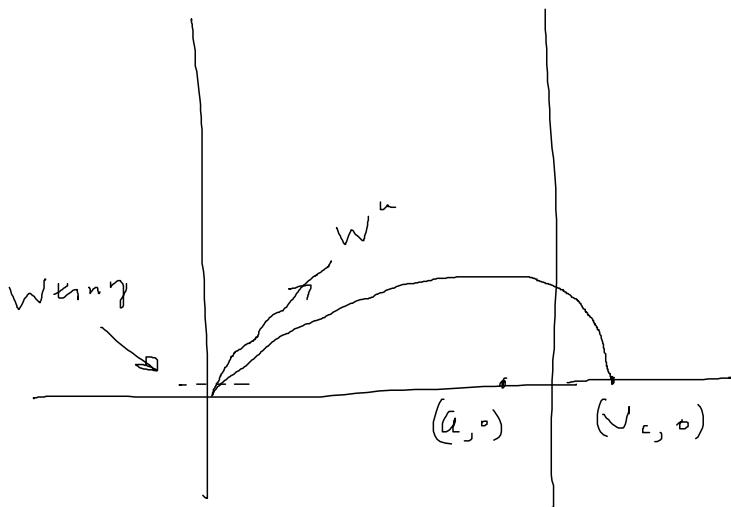
is $\begin{pmatrix} pos \\ pos \end{pmatrix}$ specifically : $\begin{pmatrix} v_t \\ w_t \end{pmatrix} = \begin{pmatrix} w \\ -\sigma w - f(v) \end{pmatrix} = \begin{pmatrix} w \\ -\sigma w \end{pmatrix}$

at any point

$$(0, w_0) \quad (w_0 > 0)$$

Q: must w^u cross the right side?

could it simply head off to the horizons?



(11)

$(v(t), w(t)) \in W^u$ and \exists

$t > -N \Rightarrow w(t) > w_{tiny}$ we know that

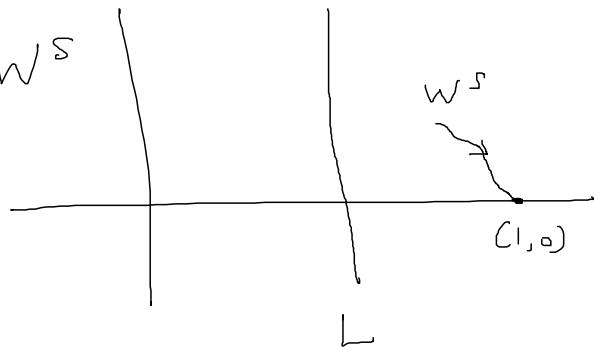
$$\frac{dv}{dt} = w(t) > w_{tiny} \Rightarrow v(t) > w_{tiny} t + v(-N)$$

$$\Rightarrow v(t) \text{ will get past } a + \varepsilon$$

in finite time.

Thus proves that W^u must intersect the line L .

Using similar arguments, W^s must intersect L ,
 (Hw!)



And so we conclude that

W^u intersects the line at a point

$$(a + \varepsilon, w_0(\sigma))$$

and W^s intersects the line at a point

$$(a + \varepsilon, w_1(\sigma))$$

Obs 1: if $\sigma = 0$ then the level sets of E contain the trajectories of the flow.

$$\text{Since } E(0, 0) = 0 < E(1, 0)$$

because $\int_0^1 f(z) dz > 0$

we know that $w_0(0) < w_1(0)$.

We now claim that for sufficiently negative

$$\sigma \quad w_1(\sigma) < w_0(\sigma)$$

then we're done by the intermediate value

theorem: $\exists \sigma_0 \ni w_0(r_0) = w_1(\sigma_0) \Rightarrow$

W^u and W^s intersect at a point \Rightarrow in the region

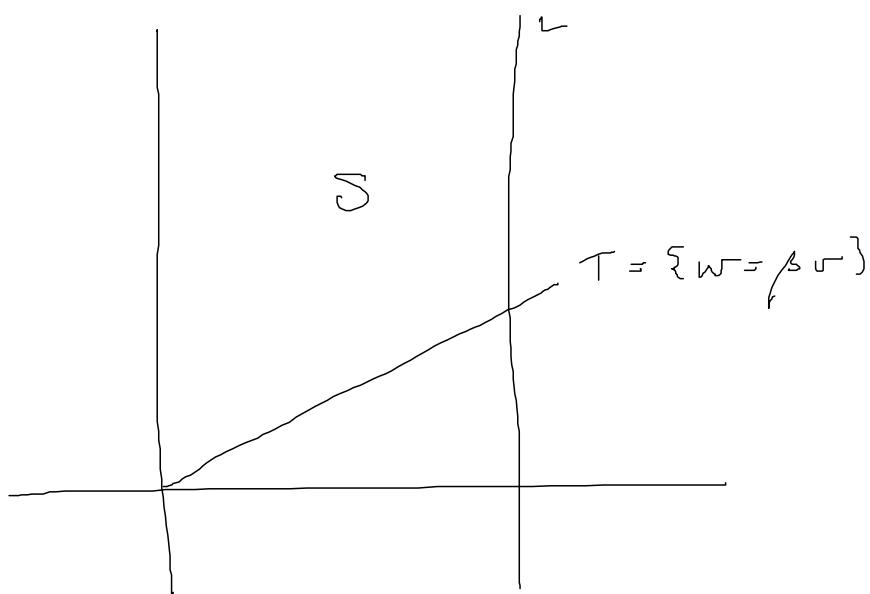
$\{r > 0, w > 0\}$ $W^u = W^s$ and we're done.

Note: w/ harder work, we can show σ_0 is unique.

and $\exists!$ speed for the travelling wave



We consider a new region in phase space.



Fix $\beta > 0$ if σ is sufficiently negative
 w^u will initially enter S . Why?

w^u near $(0,0)$ is tangent to

$$w = \lambda_+^\sigma v > \beta v$$

$$\text{If } \frac{-\sigma + \sqrt{\sigma^2 - 4f'(0)}}{2} > \beta \quad \text{this is true for} \\ \sigma < -\beta - \frac{f'(0)}{\beta}$$

So for σ suff negative w^u enters the region S

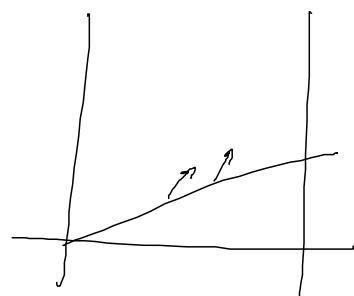
Look at the vector field on T .

$$\begin{pmatrix} \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} w \\ -vw - f(v) \end{pmatrix} = \begin{pmatrix} \beta v \\ -\sigma \beta v - f(v) \end{pmatrix}$$

I claim that we can ensure that the

flow cannot cross T .

(14)



$$\text{I.e. } \begin{pmatrix} v \\ w \end{pmatrix} \cdot \begin{pmatrix} \text{normal} \\ +T \end{pmatrix} > 0$$

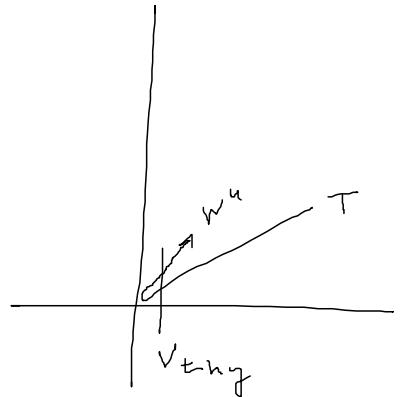
$$\begin{pmatrix} v \\ w \end{pmatrix} \cdot \begin{pmatrix} -\beta \\ 1 \end{pmatrix} > 0$$

$$\Rightarrow \text{Want } \begin{pmatrix} \beta v \\ -\sigma \beta v - f(v) \end{pmatrix} \cdot \begin{pmatrix} -\beta \\ 1 \end{pmatrix} > 0$$

$$-\beta^2 - \sigma \beta v - f(v) > 0$$

$$-\sigma \beta v > \beta^2 + f(v)$$

$$-\sigma > \frac{\beta^2 + f(v)}{\beta v}$$



want this to hold for all $v > v_{tiny}$.

since we already know that for

$$t > -N \quad (v(t), w(t)) \in S,$$

Since $f(v)$ is bounded from above, and

v is bounded from below, we know that

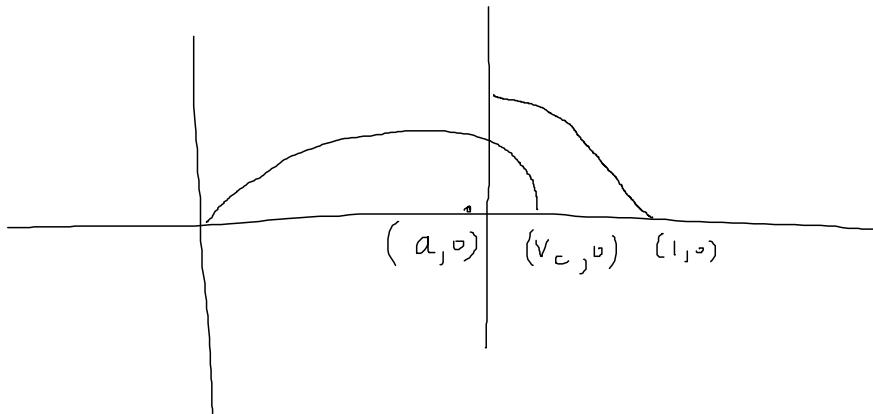
$$\frac{\beta^2 + f(v)}{\beta v} < C \quad \text{for some } C \quad \text{if } v > v_{tiny}$$

\Rightarrow by choosing or sufficiently negative,
we know W^s cannot cross the line T .

And so, $w_1(\sigma) > \beta(a + \varepsilon)$.

On the other hand, $w_1(\sigma) < w_1(0)$ for all $\sigma < 0$.

Why?



$(\varphi(t), w(t))$ are on W^s let $E(x) = E(\varphi(t), w(t))$

$$\frac{d}{dt} E(t) = -\sigma w(t)^2$$

if $\sigma = 0$ then $E(x) \equiv E(1, 0)$.

on the other hand, if $\sigma < 0$ then

$$\lim_{t \rightarrow \infty} E(t) = E(1, 0)$$

and $E(t)$ is non decreasing $\Rightarrow W^s$ is moving

"outwards" towards the level set $E(1, 0)$.

$$\Rightarrow w_1(v) < w_1(0).$$

Putting it all together,

If we take $\beta > 1$ so that

$$w_1(0) < \beta(a+\varepsilon)$$

and then take σ suff negative such that

$$\beta(a+\varepsilon) < w_0(\sigma)$$

then we've got

$$w_1(r) < w_1(\sigma) < \beta(a+\varepsilon) < w_0(\sigma)$$

and so $w_1(\sigma) < w_0(\sigma)$ as desired.

