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We continue our search for exact solutions.

Separable solutions:

Consider

$$\begin{cases} u_t = \lambda u & \text{in } \mathcal{U} \times (0, \infty) \\ u=0 & \text{on } \partial \mathcal{U} \times [0, \infty) \\ u=g & \text{on } \mathcal{U} \times \{t=0\}. \end{cases}$$

Make a separable ansatz

$$u(x, t) = v(t) w(x)$$

$$\Rightarrow v_t w = v(t) \Delta w$$

$$\Rightarrow \frac{v_t}{v} = \frac{\Delta w}{w}$$

LHS is indep of x .
RHS is indep of t

\Rightarrow both sides are indep of $x \& t$

$$\Rightarrow \frac{v_t}{v} = \frac{\Delta w}{w} = M \quad \text{for some } M \text{ true in } \mathcal{U} \times (0, \infty)$$

$$v_t = M v \quad \& \quad \Delta w = M w$$

can we solve these? well, the first one, we can.

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λ is an eigenvalue of the operator
 $-\Delta$ on U if \exists a function w , not
 identically zero, solving

$$\begin{cases} -\Delta w = \lambda w & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

if λ is an eigenvalue and w is the affiliated
 eigenfunction then

$$u(x, t) = d e^{-\lambda t} w(x)$$

solves $u_t = \Delta u$ with $u=0$ on $\partial U \times [0, \infty)$.

If you have m eigenvalues & m eigenfunctions

then $u(x, t) := \sum_{n=1}^m d_n e^{-\lambda_n t} w_n(x)$

is a solution with initial data $\sum_{n=1}^{\infty} d_n w_n(x)$.

Q: Under what conditions is there a
 countable family of eigenfunctions
 that span the space the initial data
 lies in?

$$\sum_{k=1}^{\infty} d_k w_k(x) = g(x)$$

can you choose d_k ?

Under what conditions does the formal solution

$$u(x,t) := \sum_{n=1}^{\infty} d_n e^{-\lambda_n t} w_n(x) \quad \textcircled{E}$$

make sense?

Assume \mathcal{V} is open & bounded and $\partial\mathcal{V} \in C^1$

In chapter 4, we'll find that there are countably many eigenvalues λ_n , they're all real, and the eigenfunctions w_n span $L^2(\mathcal{V})$.

In chapter 7, one finds that the representation \textcircled{E} works. And so via separable solutions, one constructs a solution to the initial value problem for any L^2 initial data.

Further, as $n \rightarrow \infty$ $\lambda_n \rightarrow \infty$. But one would like information about the rate with which $\lambda_n \rightarrow \infty$.

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§4.2 Plane wave solutions.

Given a PDE w/ $n+1$ variables
 (x_1, \dots, x_n, t)

one can make a plane wave ansatz

Seek $v : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$u(x, t) = v(\vec{x} \cdot \vec{y} - \sigma t)$$

solves the PDE. σ is the speed of the solution and \vec{y} is normal to the plane.

i.e. at each time t , u is constant on the plane with normal \vec{y} .

Consider $u_t = \Delta u$ in \mathbb{R}^n assume.

$$u(\vec{x}, t) = v(\vec{y} \cdot \vec{x} - \sigma t)$$

let $\eta := \vec{y} \cdot \vec{x} - \sigma t$ then

$$u_t = \Delta u$$

$$\Rightarrow -\sigma v'(\eta) = |\vec{y}|^2 v''(\eta)$$

$$\Rightarrow v''(\eta) = -\frac{\sigma}{|\vec{y}|^2} v'(\eta)$$

$$\Rightarrow v(\eta) = v(0) - \frac{|\vec{y}|^2}{\sigma} v'(0) \left[e^{-\frac{\sigma}{|\vec{y}|^2} \eta} - 1 \right]$$

and so

$$u(x,t) = v(0) - \frac{|y|^2}{\sigma} v'(0) \left[e^{-\frac{\sigma}{|y|^2}(y \cdot x - \sigma t)} - 1 \right]$$

Idea: if $-\frac{\sigma}{|y|^2} = i$ then the spatial structure is that of a Fourier function. (i.e. one of the basis functions for a Fourier transform)

Assume $\boxed{-\sigma = |y|^2}$

then $u(x,t) = e^{i(y \cdot x + |y|^2 t)}$ solves $u_t = \Delta u$

$$= e^{i y \cdot x - |y|^2 t}$$

$$u(x,t) = e^{-|y|^2 t} e^{iy \cdot x}$$

Q: if $u(x,t) = e^{-|y|^2 t} e^{iy \cdot x}$ solves $u_t = \Delta u$ and $y \in \mathbb{R}^n$ is arbitrary, how does this relate to our previous statement about eigenvalues and eigenfunctions?

A: The countable set of eigenvalues was for a bounded domain. Here we have an unbounded domain and a continuum of "eigenvalues".

(think Fourier series versus Fourier transform.)

Taking real and imaginary parts of

$$e^{-|y|^2 t} e^{iy \cdot x}$$

we have real solutions

$$e^{-|y|^2 t} \cos(y \cdot x)$$

and

$$e^{-|y|^2 t} \sin(y \cdot x).$$

Wave equation $u_{tt} - \Delta u = 0$

Plug & chng

$$u(x, t) = e^{i(y \cdot x \pm |y|t)}$$

is a solution Note $|u(x, t)| = 1$ (no dissipation)

Airy's equation $u_t + u_{xxx} = 0$

Plug & chng

$$u(x, t) = e^{i(yx + y^3 t)}$$

is a solution

Note: $u(x, t) = e^{iy(x + y^2 t)}$

speed of propagation depends on the frequency
of the wave

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there are waves of different frequencies travel with different speeds.

This is to be contrasted with solutions of $u_{tt} - u_{xx} = 0$

$$u(x, t) = e^{iy(x \pm t)}$$

all waves travel with speed ± 1 or -1 .

defn: If plane waves of different frequencies travel with different speeds, the PDE is called dispersive.

lesson: wave equation is dispersive

Airy equation is dispersive

Schrödinger equation is dispersive too:

$$iu_t + \Delta u = 0$$

$$\Rightarrow u(x, t) = e^{i[y \cdot x - |y|^2 t]}$$

and again, the speed depends on the frequency.

Solitons

A soliton is a very special type of travelling wave solution. They were first discovered in the KdV equation (Korteweg de Vries)

$$u_t + 6uu_x + u_{xxx} = 0 \quad \text{on } \mathbb{R} \times [0, \infty)$$

This is the (dispersive) Airy equation with nonlinearity added to it.

Seek a travelling wave solution:

$$u(x, t) = v(x - \sigma t) \quad \eta = x - \sigma t$$

Note that if $u(x, t)$ is a solution of $u_t + 6uu_x + u_{xxx} = 0$ then $u(-x, -t)$ is also a solution. Since $\eta = x - \sigma t$, this implies v is an even function of η .

$$-\sigma v' + 6v v' + v''' = 0$$

$$\Rightarrow -\sigma v' + 3(v^2)' + v''' = 0$$

$$\Rightarrow -\sigma v + 3v^2 + v'' = a \quad \leftarrow \text{Integration Constant}$$

We'll seek a solution v such that $v, v', v'' \rightarrow 0$ as $\eta \rightarrow \pm\infty$. This implies the integration constant $a = 0$.

Now, view $v'' = \sigma v - 3v^2$ as a system of first order ODE.

$$\begin{cases} v' = w \\ w' = \sigma v - 3v^2 \end{cases}$$

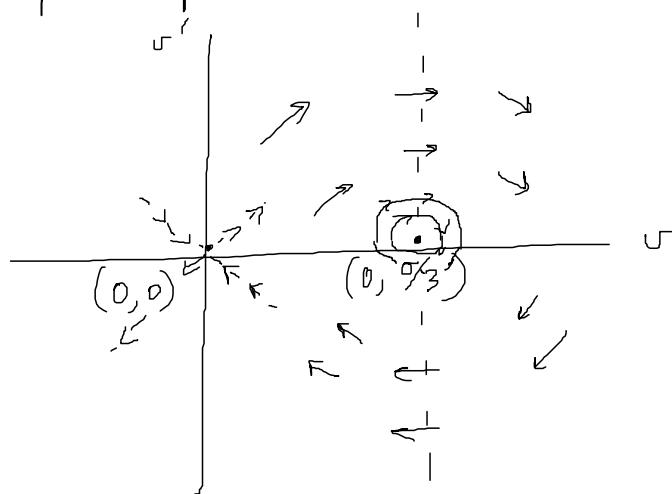
There are two fixed points $(0, 0)$ and $(\frac{\sigma}{3}, 0)$. Assume $\sigma > 0$.

Linearizing about $(0, 0)$, the eigenvalues are $\pm \sqrt{\sigma}$

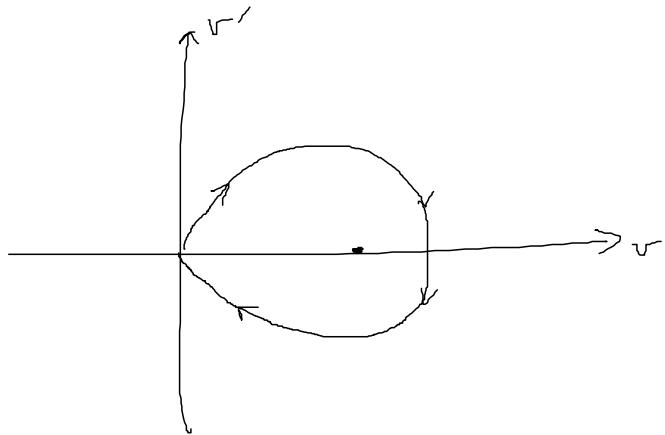
$$\begin{aligned} \text{eigenvalues} &= \begin{pmatrix} 1 \\ \sqrt{\sigma} \end{pmatrix} e^{+\sqrt{\sigma}t} \\ &\quad \begin{pmatrix} 1 \\ -\sqrt{\sigma} \end{pmatrix} e^{-\sqrt{\sigma}t} \end{aligned}$$

Linearizing about $(\frac{\sigma}{3}, 0)$, the eigenvalues are $\pm i\sqrt{-\sigma}$

Phase plane portrait



We seek a homoclinic orbit in the (v, v') phase space. That is, as $\gamma \rightarrow -\infty$ $(v(\gamma), v'(\gamma)) \rightarrow (0, 0)$ and as $\gamma \rightarrow +\infty$ $(v(\gamma), v'(\gamma)) \rightarrow (0, 0)$.



From the phase plane, it looks plausible that $v(\gamma) > 0$ and $v'(\gamma) > 0$ for $\gamma < 0$ and $v(\gamma) > 0$ and $v'(\gamma) < 0$ for $\gamma > 0$.

Fortunately, it turns out that we can explicitly solve for this homoclinic orbit.

multiply by v'

$$-\sigma vv' + 3v^2v' + v'v'' = 0$$

$$-\frac{\sigma}{2}(v^2)' + (v^3)' + \frac{1}{2}(v')^2 = 0$$

amn't we clever!

$$\Rightarrow -\frac{\sigma}{2} v^2 + v^3 + \frac{1}{2}(v')^2 = b$$

$v(\eta) \rightarrow 0$ and $v'(\eta) \rightarrow 0$
as $\eta \rightarrow \pm \infty$.

This implies $b=0$.

And so for the special solutions we seek

$$-\frac{\sigma}{2} v^2 + v^3 + \frac{1}{2}(v')^2 = 0$$

$$\Rightarrow -\sigma v^2 + 2v^3 + (v')^2 = 0$$

$$\Rightarrow v' = \pm \sqrt{\sigma v^2 - 2v^3}$$

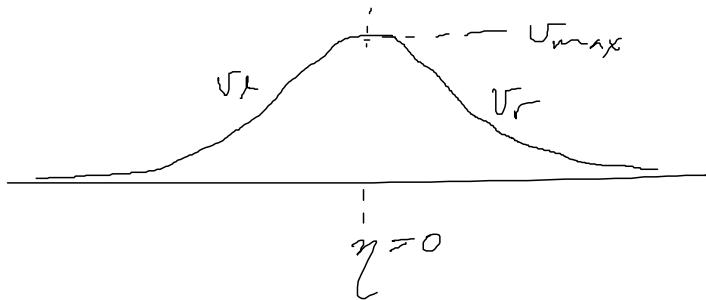
From the phase plane, $v(\eta) > 0$

$$\Rightarrow v' = \pm v \sqrt{\sigma - 2v^2}$$

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$v(\eta) \rightarrow 0$ at $\pm\infty$ and $v > 0$.

Seek a solution



that is $v' > 0$ for $\eta < 0$ and $v' < 0$ for $\eta > 0$.

That is, I want to solve

$$v'(\eta) = \begin{cases} \sqrt{\sigma - 2v} & \text{on } (-\infty, 0) \\ -\sqrt{\sigma - 2v} & \text{on } (0, \infty) \end{cases}$$

Note that $v(\eta)$ is invertible on each "leg".

$$v_L : (-\infty, 0] \rightarrow (0, v_{\max}]$$

$$v_R : [0, \infty) \rightarrow (0, v_{\max}]$$

each is invertible. Let $\eta_L : (0, v_{\max}] \rightarrow (-\infty, 0]$

$$\eta_R : (0, v_{\max}] \rightarrow [0, \infty)$$

be the inverse

then $\eta_\ell(v_\ell(\eta)) = \eta$ on $(-\infty, 0]$
 $\eta_r(v_r(\eta)) = \eta$ on $[0, \infty)$.

$$\text{From } v \circ \eta_\ell(v_\ell(\eta)) = \eta$$

$$\Rightarrow \frac{d\eta_\ell}{dv}(v_\ell(\eta)) \frac{dv_\ell}{d\eta}(\eta) = 1 \quad \text{on } (-\infty, 0]$$

$$\Rightarrow \frac{d\eta_\ell}{dv}(v_\ell(\eta)) \left[v_\ell(\eta) \sqrt{\sigma - 2v_\ell(\eta)} \right] = 1 \quad \text{on } (-\infty, 0]$$

$$\Rightarrow \frac{d\eta_\ell}{dv}(v) \left[v \sqrt{\sigma - 2v} \right] = 1 \quad \text{on } (0, v_{\max}]$$

$$\Rightarrow \frac{d\eta_\ell}{dv} = \frac{1}{v \sqrt{\sigma - 2v}} \quad \text{on } (0, v_{\max}]$$

you can solve this explicitly!

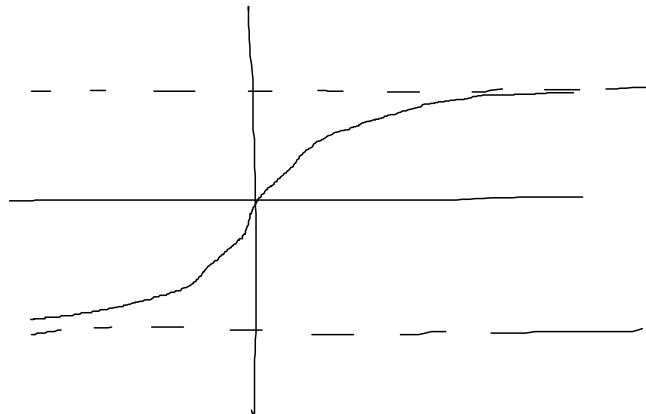
$$\eta_\ell(v) = -\frac{2}{\sigma} \operatorname{arctanh} \left(\sqrt{\frac{\sigma - 2v}{\sigma}} \right) + C$$

Similarly,

$$\eta_r(v) = \frac{2}{\sigma} \operatorname{arctanh} \left(\sqrt{\frac{\sigma - 2v}{\sigma}} \right) + d$$

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tanh:



$$\Rightarrow \operatorname{arctanh} : (-1, 1) \rightarrow (-\infty, \infty)$$

as $\sqrt{\frac{\sigma - 2v}{\sigma}} \rightarrow 1 \quad \eta_r(v) \rightarrow \infty$

as $\sqrt{\frac{\sigma - 2v}{\sigma}} \rightarrow 0 \quad \eta_r(v) \rightarrow d.$

lets take $c=0$ and $d=0$.

we see that as $v \downarrow 0 \quad \eta_r(v) \rightarrow \infty$

and as $v \uparrow 0 \quad \eta_r(v) \rightarrow 0$

similarly, as $v \downarrow 0 \quad \eta_e \rightarrow -\infty$

and as $v \uparrow 0 \quad \eta_e(v) \rightarrow 0$.

$$\Rightarrow v_{\max} = \sigma/2$$

we have $\eta_e(v)$ and $\eta_r(v)$

we want $v(\gamma) \quad v : (-\infty, \infty) \rightarrow [0, v_{\max}]$

So we invert γ_L and γ_R .

$$\gamma_L(v) = -\frac{2}{\sigma} \operatorname{arctanh}\left(\sqrt{\frac{\sigma-2v}{\sigma}}\right) \quad \text{on } [0, v_{\max}]$$

$$\Rightarrow v(\gamma) = \frac{\sigma}{2} \operatorname{sech}^2\left(\frac{\sqrt{\sigma}}{2} \gamma\right) \quad \text{on } (-\infty, 0]$$

$$\gamma_R(v) = \frac{2}{\sigma} \operatorname{arctanh}\left(\sqrt{\frac{\sigma-2v}{\sigma}}\right) \quad \text{on } (0, v_{\max}]$$

$$\Rightarrow v(\gamma) = \frac{\sigma}{2} \operatorname{sech}^2\left(\frac{\sqrt{\sigma}}{2} \gamma\right) \quad \text{on } [0, \infty)$$

We've found our solution!

$$v : (-\infty, \infty) \rightarrow [0, v_{\max}]$$

$$v(\gamma) = \frac{\sigma}{2} \operatorname{sech}^2\left(\frac{\sqrt{\sigma}}{2} \gamma\right) \quad v_{\max} = \frac{\sigma}{2}$$

$$\Rightarrow u(x, t) = \frac{\sigma}{2} \operatorname{sech}^2\left(\frac{\sqrt{\sigma}}{2} (x - \sigma t)\right)$$

Note: the taller the soliton, the faster it moves. Since

$$\sigma = \text{speed} = 2v_{\max}.$$