What do we have so far w/ these conservation laws

$$u_t + (F(u))_x = 0$$

where $u : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$?

(1) with C^2 initial data, we know how to construct solutions via characteristics. There are classical solutions that can lose regularity in finite time. That is, they're C^2 on $\mathbb{R} \times (0, T)$ but become discontinuous at time T.

(2) with piecewise constant initial data, we can carefully construct functions $u(x,t)$ via the method of characteristics but we don't know that these constructed objects are integral solutions unless the additional constraint

$$[[F(u)]] = \sigma [[u]]$$

across each curve of discontinuity in u.

(3) We now see how to construct integral solutions but find that they're not unique. So we introduce a selection mechanism.

selection mechanism:

"the entropy condition"

across any curve of discontinuity

$$F'(u_\ell) > \sigma > F'(u_r)$$

where u_ℓ is the limit from the left, u_r is the limit from the right, and σ is the speed of the curve

---

A curve of discontinuity that satisfies both the RH condition + the entropy condition is called a shock.

Oleinik introduced an entropy condition specifically for the case when F is uniformly convex ($F''(y) \geq \theta > 0 \quad \forall y \in \mathbb{R}$.)
The entropy condition "tests" u everywhere, not just at discontinuity.

u satisfies the Oleinik entropy condition if $\exists\ C > 0$, $C < \infty$ such that for all x and $z > 0$ and all t,

$$u(x+z,t) - u(x,t) \leq \frac{C}{t} z. \quad \circledast$$

Note that ⊛ implies that the function

$$x \to u(x,t) - \frac{c}{t}x$$

is non increasing. Why? Let $y > x$.

Then $y = x + z$ some $z > 0$.

$$u(x+z,t) - \frac{c}{t}(x+z) \le u(x,t) - \frac{c}{t}x$$

Since $u(x+z,t) - u(x,t) \le \frac{c}{t}z$

Therefore the function $x \to u(x,t) - \frac{c}{t}x$ has left hand and right-hand limits at each point. It then follows that $x \to u(x,t)$ has lefthand and righthand limits at each point with $u_\ell(x,t) \ge u_r(x,t)$. If F is strictly convex, it follows that $F'(u_\ell) > F'(u_r)$

at any point of discontinuity, which was the original entropy condition

What sorts of results do we have for

$$u_t + (F(u))_x = 0?$$

The problem we want to solve is

$$\begin{cases} u_t + (F(u))_x = 0 & \text{on } \mathbb{R} \times (0,\infty) \\ u = g & \text{on } \mathbb{R} \times \{t=0\} \end{cases}$$

Given g, let $h(x) := \int_0^x g(y)\,dy$ and consider the auxiliary problem

$$\textcircled{\star} \quad \begin{cases} w_t + F(w_x) = 0 & \text{on } \mathbb{R} \times (0,\infty) \\ w = h & \text{on } \mathbb{R} \times \{t=0\} \end{cases}$$

If $\textcircled{\star}$ has a smooth solution w then

$$u(x,t) := \frac{\partial}{\partial x} w(x,t)$$

will solve the hyperbolic conservation law problem we care about.

Q. Do we know that $\textcircled{\star}$ has a smooth solution?

In §3.3, Evans considers Hamilton-Jacobi equations: $w_t + F(Dw, x) = 0$ on $\mathbb{R}^n \times (0,\infty)$

These equations have characteristic equations

$$\begin{cases} \dfrac{d\vec{x}}{ds} = D_p F(\vec{p}, \vec{x}) \\ \dfrac{d\vec{p}}{ds} = -D_x F(\vec{p}, x) \end{cases}$$

These ODEs have a Hamiltonian structure. They can be studied by methods from classical mechanics. Specifically, there's a Lagrangian related to the Hamiltonian and one can introduce an <u>action functional</u>. Solutions of the characteristics are then related to the variational problem of minimizing the action. In short, given a Hamilton-Jacobi PDE

$$w_t + F(Dw, x) = 0$$

one can write a weak solution explicitly in terms of a variational problem. This is done in the book for the special case when F is independent of x, $p \to F(p)$ is convex, and F is superlinear: $\lim_{|p| \to \infty} \dfrac{F(p)}{|p|} = \infty$

In this case, one introduces the Legendre transform of F:

<u>defn</u>: the Legendre transform of F is

$$F^* : \mathbb{R}^n \to \mathbb{R}$$

$$F^*(\vec{p}) := \sup_{\vec{q} \in \mathbb{R}^n} \{ \vec{p} \cdot \vec{q} - F(\vec{q}) \}$$

Given the Legendre transform of F one then defines

$$w(x,t) := \min_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + h(y) \right\}$$

where $L = F^*$. By §3.3 w is Lipschitz continuous, is differentiable almost everywhere in $\mathbb{R}^n \times (0,\infty)$ and solves the initial value problem

$$\begin{cases} w_t + F(Dw) = 0 & \text{a.e. in } \mathbb{R}^n \times (0,\infty) \\ w = h & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

So we have a solution for the w equation. This makes us guess

$$u(x,t) := \frac{\partial}{\partial x} \min_{y \in \mathbb{R}} \left\{ tL\left(\frac{x-y}{t}\right) + h(y) \right\}$$

where $L = F^*$

Does this work? In fact, it works

This is the Lax-Oleinik formula for u, the solution of $u_t + (F(u))_x = 0$.

Theorem (Lax-Oleinik) Assume $F: \mathbb{R} \to \mathbb{R}$ is smooth, uniformly convex ($F''(y) \geq \theta > 0 \ \forall y \in \mathbb{R}$), and superlinear ($\lim_{|p| \to \infty} \frac{F(p)}{|p|} = \infty$), and $g \in L^\infty(\mathbb{R})$. Let $h(x) := \int_0^x g(y)\,dy$. Then

i) For each $t > 0$, there exists for all but countably many $x \in \mathbb{R}$ a unique $y(x,t)$ such that

$$tL\left(\frac{x - y(x,t)}{t}\right) + h(y(x,t)) = \min_{y \in \mathbb{R}} \left\{ tL\left(\frac{x-y}{t}\right) + h(y) \right\}$$

where L is the Legendre transform of F.

ii) $x \to y(x,t)$ is nondecreasing

iii) Let $G := (F')^{-1}$. For each $t > 0$

$$u(x,t) := G\left(\frac{x - y(x,t)}{t}\right)$$

is an integral solution of

$$\begin{cases} u_t + (F(u))_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{in } \mathbb{R} \times \{t = 0\} \end{cases}$$

iv) $\exists\, C>0$ such that u satisfies the Lax-Oleinik entropy condition: for all $x\in\mathbb{R}$, $\forall t>0$, $\forall z>0$

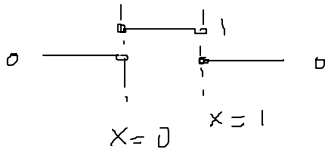
$$u(x+z,t)-u(x,t)\le \frac{C}{t}z$$

v) any integral solution of $\begin{cases} u_t+(F(u))_x=0 \\ u=g \end{cases}$ that satisfies the Oleinik entropy condition must be this entropy solution. (Uniqueness of entropy solutions.)

The theorem is proven using methods from §3.3 and so we won't prove the theorem. The Lax-Oleinik formula for the solution u is very useful. It allows us to prove two asymptotic in time results.

Theorem: Assume F is smooth, uniformly convex, $F(0)=0$, and superlinear. Assume $g\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$. Then there exists $C<\infty$ so that the entropy solution satisfies $|u(x,t)|\le \frac{C}{\sqrt{t}}$ for all $x\in\mathbb{R}$ $\forall t>0$.

Note: we already saw such decay for the rarefaction solution of $u_t + (\frac{1}{2}u^2)_x = 0$ with initial data

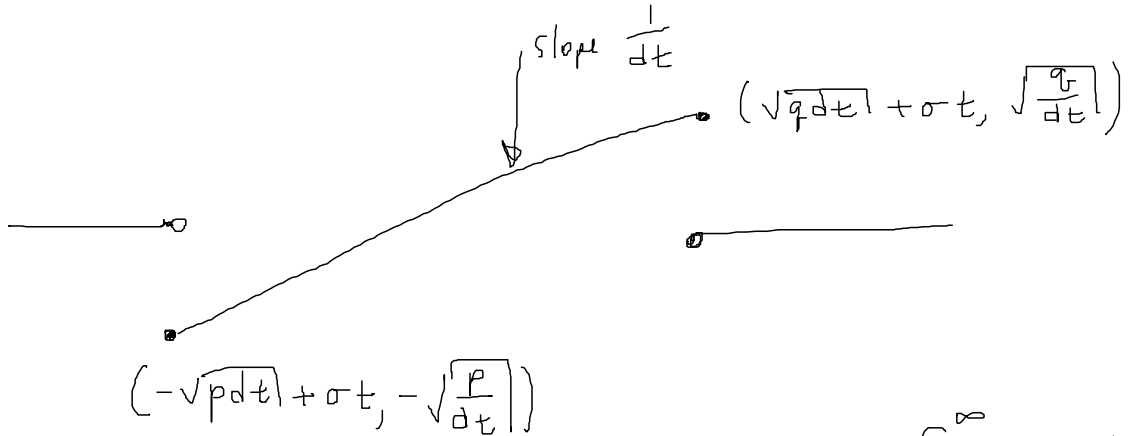


there, we saw that the height decayed like $\frac{1}{t}$ which is even faster than $\frac{1}{\sqrt{t}}$.

If the initial data g is compactly supported, one can say even more about the asymptotic behavior of u. To do this, we introduce

N-waves. Fix $p>0$, $q>0$, $d>0$. Let

$$N(x,t) := \begin{cases} \frac{1}{d}\left(\frac{x}{t} - \sigma\right) & \text{if } -\sqrt{pdt} \le x - \sigma t \le \sqrt{qdt} \\ 0 & \text{otherwise} \end{cases}$$



$$\int_{-\infty}^{\infty} N(x,t)\,dx = \frac{q-p}{2}$$

Theorem: (Asymptotics in L^1).

Assume F is C^3 and

$g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and has compact support. Then $\exists!$ N-wave

And $\exists\, C < \infty$ such that the entropy solution u satisfies

$$\|u(\cdot,t) - N(\cdot,t)\|_{L^1} \le \frac{C}{\sqrt{t}}$$

for all $t > 0$.

How is this N-wave found? Need σ, d, p, q

The flux function F determines σ and d.

$$\sigma := F'(0)$$

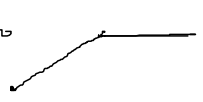
$$d := F''(0) > 0.$$

The initial data g determines p and q.

$$p := -2 \min_{y \in \mathbb{R}} \int_{-\infty}^{y} g(x)\,dx$$

$$q := 2 \max_{y \in \mathbb{R}} \int_{y}^{\infty} g(x)\,dx.$$

Note that if $g \ge 0$ then $p = 0 \Rightarrow$ N wave is

if $g \le 0$ then $q = 0 \Rightarrow$ N wave is 

Should we be excited about these results? We've found integral solutions that exist for all time and satisfy the entropy condition. And they don't even require nice initial data!

Sounds great, but did you notice that nothing was said about how smooth/not smooth the solutions are? And what if you wanted to compute solutions on a computer?

Fortunately, there is another way to access the entropy solution -- via Riemann problems.

Riemann Problem

Consider $u_t + (F(u))_x = 0$

with $g(x) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$

that is, piecewise-constant initial data.

Theorem:

Assume F is C^2 and is uniformly convex and let $G := (F')^{-1}$.

1) If $u_\ell > u_r$ then the unique entropy solution of the Riemann problem is

$$u(x,t) = \begin{cases} u_\ell & \text{if } \frac{x}{t} < \sigma \\ u_r & \text{if } \frac{x}{t} > \sigma \end{cases} \qquad x \in \mathbb{R},\ t > 0$$

where $\sigma := \dfrac{F(u_\ell) - F(u_r)}{u_\ell - u_r}$

2) If $u_\ell < u_r$ then the unique entropy solution of the Riemann problem is

$$u(x,t) = \begin{cases} u_\ell & \text{if } \frac{x}{t} < F'(u_\ell) \\ G\left(\frac{x}{t}\right) & \text{if } F'(u_\ell) < \frac{x}{t} < F'(u_r) \\ u_r & \text{if } \frac{x}{t} > F'(u_r) \end{cases} \qquad x \in \mathbb{R},\ t > 0.$$

In case 1) there's a shock wave moving with constant speed σ. In 2) there's a rarefaction wave connecting u_ℓ and u_r

Proof:

1) Assume $u_l > u_r$. By construction, $u(x,t)$ satisfies the Rankine-Hugoniot condition and so it is an integral solution of the PDE. It remains to show that it satisfies the entropy condition.

Since $F'' \geq \theta > 0 \Rightarrow F'$ is increasing

$$\Rightarrow \quad F'(u_r) < \sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r} = \int_{u_r}^{u_l} F'(r)\,dr < F'(u_l)$$

This shows that $u(x,t)$ satisfies the entropy condition, as desired.

2) Assume $u_l < u_r$.

First, we need to check that the u is a solution in the region $F'(u_l) < \frac{x}{t} < F'(u_r)$

When does $u(x,t) = v\left(\frac{x}{t}\right)$ solve the equation?

$$u_t + (F(u))_x = -\frac{x}{t^2} v'\left(\frac{x}{t}\right) + F'\left(v\left(\frac{x}{t}\right)\right) \frac{1}{t} v'\left(\frac{x}{t}\right)$$

$$= v'\left(\frac{x}{t}\right)\left[F'\left(v\left(\frac{x}{t}\right)\right) - \frac{x}{t}\right]\frac{1}{t}$$

$$= 0 \quad \text{if } F'\left(v\left(\frac{x}{t}\right)\right) = \frac{x}{t}$$

$$\Rightarrow v\left(\frac{x}{t}\right) = (F')^{-1}\left(\frac{x}{t}\right) = G\left(\frac{x}{t}\right)$$

Is u continuous?

as $\frac{x}{t} \to F'(u_\ell)$

$$(F')^{-1}\left(\frac{x}{t}\right) \to u_\ell \quad \Rightarrow \quad v\left(\frac{x}{t}\right) \to u_\ell$$

$\Rightarrow u(x,t)$ is continuous on $\mathbb{R} \times (0,\infty)$ and solves the PDE almost everywhere (not classically at $\frac{x}{t} = F'(u_\ell)$ or $\frac{x}{t} = F'(u_r)$ need to check it's an integral solution)

It remains to show that u satisfies the Oleinik entropy condition. $\exists\, C \geq 0,\ C < \infty$ such that for all x and $z \geq 0$ and all $t > 0$

$$u(x+z,t) - u(x,t) \leq \frac{C}{t} z. \quad (\ast)$$

And so we see that we just want to show that $\exists C$ such that

$(\ast\ast)$ $$G\left(\frac{x+z}{t}\right) - G\left(\frac{x}{t}\right) \leq \frac{C}{t} z$$

for all x and t and all $z \geq 0$. This is automatically true if G is Lipschitz. The LHS $= 0$ and so $(\ast\ast)$ is automatically true if

$$\frac{x+z}{t} < F'(u_\ell) \quad \text{or} \quad \frac{x}{t} > F'(u_r)$$

So it suffices to show (**) is true for

$$F'(u_\ell) < \frac{x}{t} < \frac{x+z}{t} < F'(u_r)$$

So really, we just need that G is Lipschitz on the interval $[F'(u_\ell), F'(u_r)]$.

We know that

$$F'(G(z)) = z \quad \text{for all } z$$

$$\Rightarrow \quad F''(G(z))\, G'(z) = 1$$

$$\Rightarrow G'(z) = \frac{1}{F''(G(z))}$$

since $F''(w) \geq \theta > 0$ for all w, we see that G' is differentiable. and hence is Lipschitz on a bounded interval. This proves that u satisfies the entropy condition, as desired. //

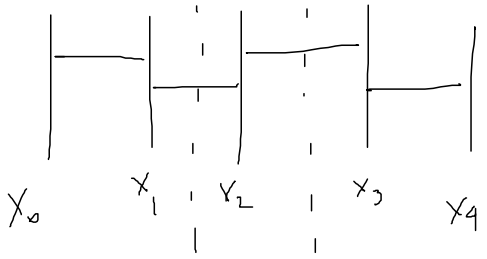
Who cares!?! Why do we want to construct entropy solutions for such initial data? By the uniqueness theorem, we already know they existed.

ans 1: these new representations are explicit

ans 2: you can use them to construct the other solutions.

Cartoon idea.

Given $g \in L^\infty(\mathbb{R})$, approximate g with a piecewise constant function g_N (N steps.) Using the exact solutions we've just looked at, you can take the g_N initial data and "time step it forward" by a time Δt. (Choose Δt small enough so that it's not clear that the initial data wasn't just a step function.



take Δt small enough so that the solution between $x_1 + \frac{\Delta x}{2}$ and $x_2 + \frac{\Delta x}{2}$ doesn't "feel" the initial data on $[x_0, x_1]$ or $[x_3, x_4]$

Now approximate your $u(x,t)$ with piecewise constant functions and advance your time step again.

Theorem: (Ghimm.) as $\Delta t \to 0$ and $N \to \infty$ the converges to an entropy solution with init. cl data g.

It's a scheme that can be implemented on a computer, unlike the implicit representation given earlier in terms of G.

Also, this type of approach works for systems of conservation laws, which are much much much much harder than single conservation laws.