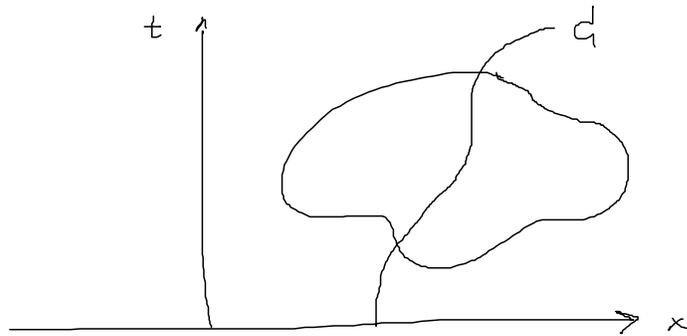


①

Consider an integral solution of $u_t + (F(u))_x = 0$ which is smooth in V_L and $V_R \subseteq \mathbb{R}^2$, both open



and $\partial V_L \cap \partial V_R = \emptyset$. Assume u is C^1 in V_L and in V_R

① Let v be any test function with compact support in V_L . Then

$$0 = \int_0^\infty \int_{-\infty}^\infty u v_t + F(u) v_x \, dx \, dt \quad \text{since } u \text{ is an integral solution}$$

$$= - \int_0^\infty \int_{-\infty}^\infty u_t v + (F(u))_x v \, dx \, dt \quad \text{since } v \text{ has compact support in } V_L, \text{ which is open. (didn't pick up any boundary terms from } t=0.)$$

$$\textcircled{2} \quad 0 = - \int_0^\infty \int_{-\infty}^\infty [u_t + (F(u))_x] v \, dx \, dt$$

③ v true for all test functions with support

in V_L . And we've assumed u is $C^1(V_L) \Rightarrow$

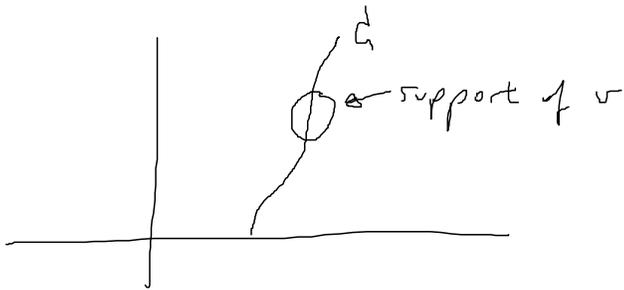
$u_t + (F(u))_x$ is continuous. \therefore from $*$, we conclude

$$u_t + (F(u))_x = 0 \quad \text{on } V_L.$$

②

Which is to say that u is a classical solution of the PDE in V_L . Similarly, u is a classical solution of the PDE in V_R .

Q: what if the test function had support which intersected $\partial V_R \cap \partial V_L = C$?



Since u is an integral solution,

$$0 = \int_0^{\infty} \int_{-\infty}^{\infty} u v_t + F(u) v_x = 0$$

$$= \iint_{V_L} u v_t + F(u) v_x \, dx \, dt + \iint_{V_R} u v_t + F(u) v_x \, dx \, dt$$

$$\iint_{V_L} u v_t + F(u) v_x \, dx \, dt = \iint_{V_L} \begin{pmatrix} F(u) \\ u \end{pmatrix} \cdot \nabla v \, dx \, dt$$

$$= - \iint_{V_L} \operatorname{div} \begin{pmatrix} F(u) \\ u \end{pmatrix} v \, dx \, dt + \int_{\partial V_L} v \begin{pmatrix} F(u) \\ u \end{pmatrix} \cdot \nu \, dS$$

$$= - \iint_{V_L} v [(F(u))_x + u_t] \, dx \, dt + \int_{\partial V_L} v \begin{pmatrix} F(u) \\ u \end{pmatrix} \cdot \nu \, dS$$

here

$$u_L(x,t) := \lim_{\substack{(x,t) \rightarrow c \\ (x,t) \in V_L}} u(x,t)$$

(3)

assume $C = \text{supp}(u) \cap (\partial V_L \cap \partial V_R)$

then

$$\iint_{V_L} u v_t + F(u) v_x \, dx \, dt = \int_C u \begin{pmatrix} F(u) \\ u \end{pmatrix} \cdot \nu \, dl$$

since

$$\iint_{V_L} u [u_t + (F(u))_x] \, dx \, dt = 0$$

where ν is the unit normal pointing from V_L into V_R . (the outward normal.)

Similarly,

$$\iint_{V_R} u v_t + F(u) v_x \, dx \, dt = - \int_C u \begin{pmatrix} F(u) \\ u \end{pmatrix} \cdot \nu \, dl$$

where $u_R(x,t) := \lim_{\substack{(x,t) \rightarrow (x_0, t_0) \in C \\ (x,t) \in V_R}} u(x,t)$

The negative sign is because ν continues to be the unit normal pointing out of V_L .

Since $\iint_{V_L} + \iint_{V_R} = 0$, we have

$$\int_C u \begin{pmatrix} F(u_L) - F(u_R) \\ u_L - u_R \end{pmatrix} \cdot \nu \, dl = 0$$

And so

$$0 = \int_C \begin{pmatrix} F(u_L) - F(u_r) \\ u_L - u_r \end{pmatrix} \cdot \nu \, ds$$

can we conclude from this that

$$\begin{pmatrix} F(u_L) - F(u_r) \\ u_L - u_r \end{pmatrix} \cdot \nu = 0 \text{ at each point on the curve?}$$

Do we know that $\begin{pmatrix} F(u_L) - F(u_r) \\ u_L - u_r \end{pmatrix} \cdot \nu$ is continuous along the curve?

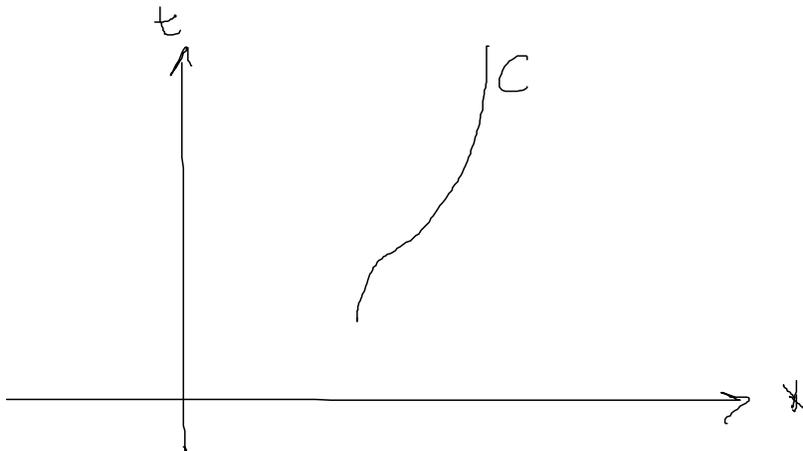
We know $u \in C^1$ on $V_L \Rightarrow u_L \in \text{continuous on } \overline{V_L}$

$u \in C^1$ on $V_R \Rightarrow u_r \in \text{continuous on } \overline{V_R}$

$\Rightarrow F(u_r), F(u_L), u_L,$ and u_r are all continuous on C .

and so we conclude

$$\begin{pmatrix} F(u_L) - F(u_r) \\ u_L - u_r \end{pmatrix} \cdot \nu = 0 \text{ at each point on the curve separating } V_L \text{ from } V_R$$



Suppose $C = \{(s(t), t)\}$
for some smooth functions.

5

then unit tangent to C is $\begin{pmatrix} \dot{s} \\ 1 \end{pmatrix} \frac{1}{\sqrt{1+(\dot{s})^2}}$

$$\text{and } \nu = \frac{1}{\sqrt{1+(\dot{s})^2}} \begin{pmatrix} 1 \\ -\dot{s} \end{pmatrix}$$

$$\text{and so } \begin{pmatrix} F(u_l) - F(u_r) \\ u_l - u_r \end{pmatrix} \cdot \nu = 0$$

$$\Rightarrow (F(u_l) - F(u_r)) = (u_l - u_r) \dot{s}$$

notation:

$$\begin{cases} [[u]] = u_l - u_r = \text{jump in } u \text{ across } C \\ [[F(u)]] = F(u_l) - F(u_r) = \text{jump in flux across } C \\ \sigma = \dot{s} = \text{speed of } C \end{cases}$$

defn: the Rankine-Hugoniot condition is

$$[[F(u)]] = \sigma [[u]]$$

the jumps and the speed may vary along C
but the relation \uparrow must always hold if the
solution is an integral solution.

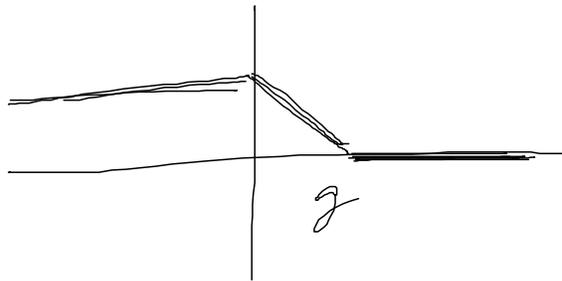
ex: Burger's equation

(6)

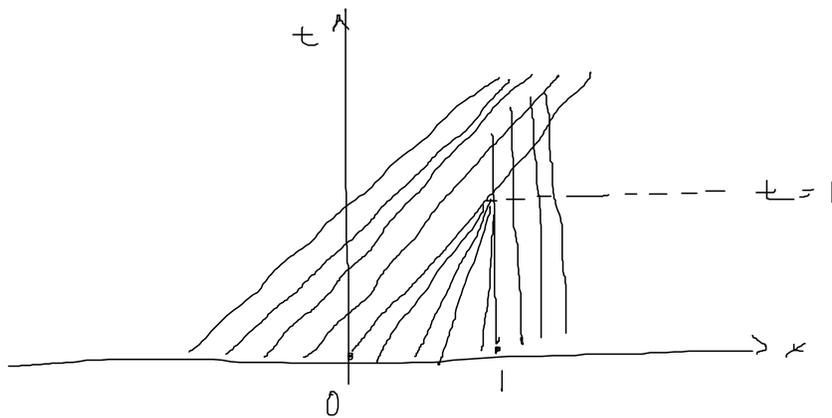
$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t=0\} \end{cases}$$

Assume initial data

$$g(x) = \begin{cases} 1 & x \leq 0 \\ 1-x & 0 \leq x \leq 1 \\ 0 & x \geq 1 \end{cases}$$



drawing the x characteristics for the problem.

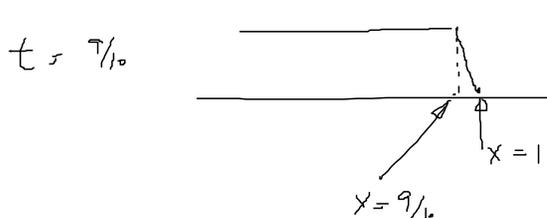
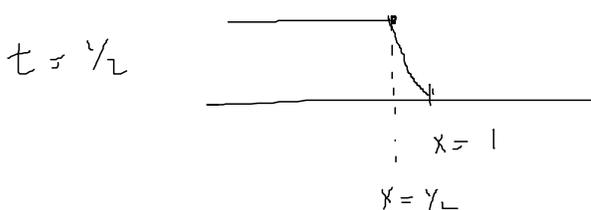
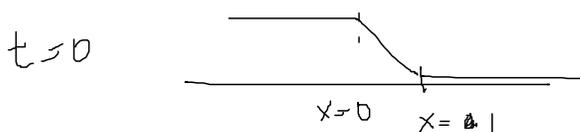


get solution

$$u(x,t) = \begin{cases} 1 & \text{if } x \leq t \text{ and } 0 \leq t \leq 1 \\ \frac{1-x}{1-t} & \text{if } t \leq x \leq 1 \text{ and } 0 \leq t \leq 1 \\ 0 & \text{if } x \geq 1 \text{ and } 0 \leq t \leq 1 \end{cases}$$

can't define the solution via characteristics beyond $t=1$!

What are the people doing?

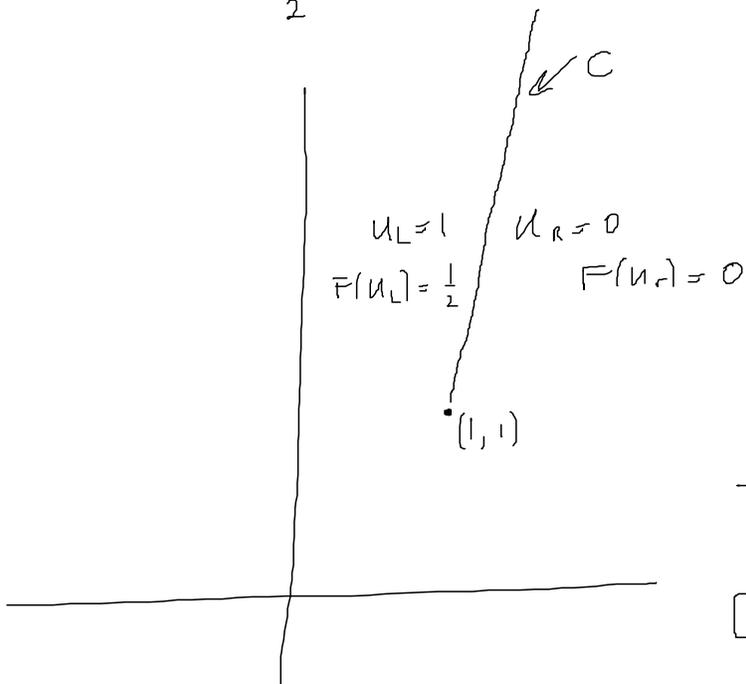


The front gets steeper and steeper

What to do at time 1?

let $s(t) = \frac{1+t}{2}$

$\dot{s} = \frac{1}{2}$



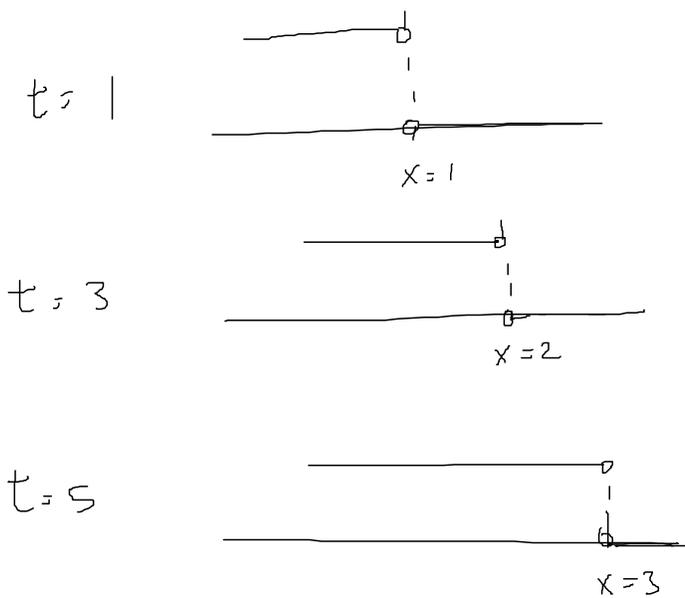
Q: does Rankine Hugoniot hold?

$[[F(u)]] \stackrel{?}{=} \dot{s} [[u]]$

$\frac{1}{2} \stackrel{?}{=} \left(\frac{1}{2}\right)(1) \checkmark$

∴ we have a valid integral solution!

The point is that we used characteristics to define the solution up to time $t < 1$. Now, we define the solution for $t \in [1, \infty)$ and since it satisfies the R-H condition we know that even though the solution is discontinuous, it's an integral solution.

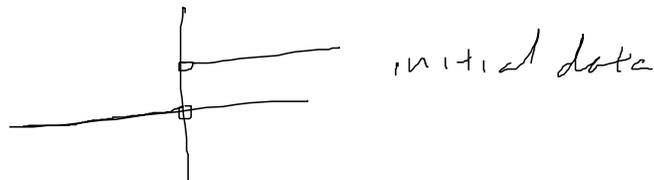


and the discont. is moving to the right with speed $\frac{1}{2}$

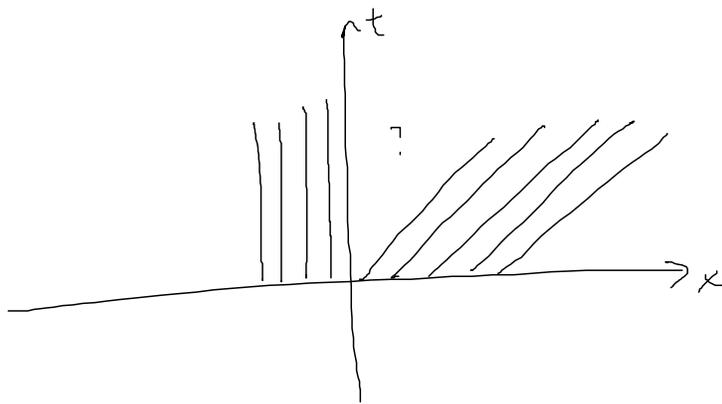
Another example w/ Burgers eqn

$$\text{Consider } \begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t=0\} \end{cases}$$

where $g(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$



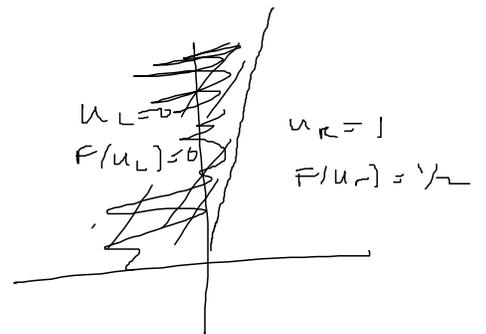
Applying the method of characteristics



clearly $u(x,t) = 0$ if $x < 0$
 $= 1$ if $x > t$

but what should we do for $u(x,t)$
 if $0 < x < t$?

plan 1 $u_1(x,t) = \begin{cases} 0 & x < t/2 \\ 1 & x > t/2 \end{cases}$



$s(t) = t/2 \quad \dot{s} = 1/2$

$[[F(u)]] = 0 - 1/2 = -1/2$
 $[[u]] = 0 - 1 = -1$

$[[F(u)]] \stackrel{?}{=} \dot{s} [[u]] \quad \checkmark$

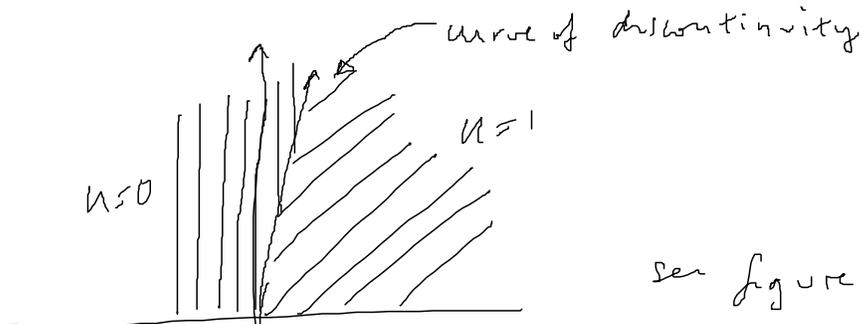
so this is a valid integral solution.

On the other hand,

$$u_2(x,t) = \begin{cases} 1 & x > t \\ \frac{x}{t} & 0 < x < t \\ 0 & x < 0 \end{cases}$$

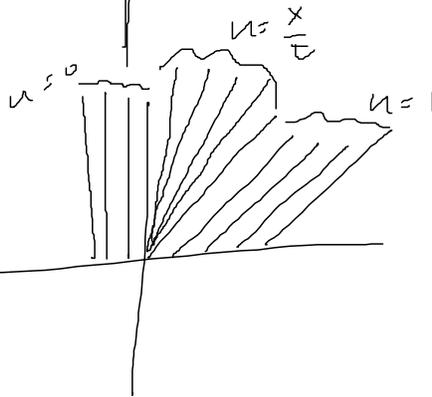
is a continuous integral solution

solution u_1



see figure on page 142

solution u_2



Yikes! we have two integral solutions to the same initial value problem! What to do? Is there some way of selecting the "preferred" solution?

Idea! we would like solutions to be well-behaved, at least backwards in time. This would mean that if $(x,t) \in \mathbb{R} \times (0, \infty)$ and you look at the characteristic through (x,t) then for previous points in time there's no disaster.

Specifically, we want to be able to exclude solutions u_1 , for which the characteristics are emerging from the shock

recall that the x -characteristics are given by the initial data and the form of the flux function:

$$\begin{cases} t(s) = s \\ x(s) = F'(g(x^*))s + x_0 \end{cases}$$

to exclude solutions u_1 we want that x -characteristics to the left are moving right with higher speed than characteristics to the right of C .

i.e. $F'(u_l) > \sigma > F'(u_r)$

* $F'(u_l) > \sigma > F'(u_r)$

1) the entropy condition. If a curve of discontinuity C satisfies both the Rankine-Hugoniot condition and the entropy condition, it's called a shock.

Note: if $F'' \geq 0 > 0$ then F' is strictly increasing and the entropy condition reduces to $u_l > u_r$

another example:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

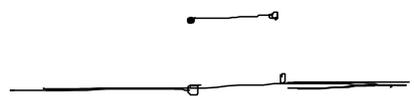
$$F(z) = \frac{1}{2} z^2$$

$$F'(z) = z$$

$$F''(z) = 1$$

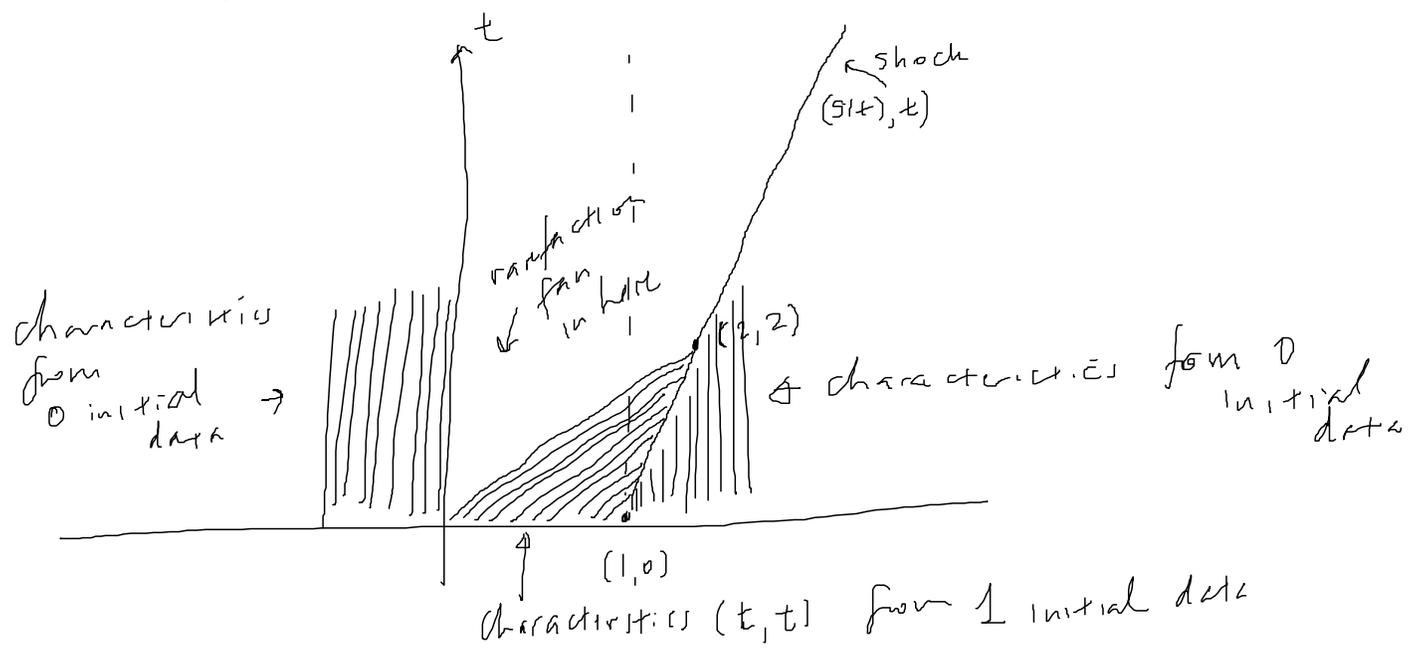
entropy condition $\Rightarrow u_l > u_r$.

consider initial data $g(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leq x \leq 1 \\ 0 & 1 < x \end{cases}$



We'll have a shock starting from $(1, 0)$. It will be $S(t) = \frac{1}{2}t + 1$

Starting at $(0, 0)$, we'll have a rarefaction fan



Note that at the point (2,2) the shock no longer has $u_L \equiv 1$ and $u_R \equiv 0$. At this point, the solution from the initial data $g \equiv 1$ on $[0,1]$ has been all "swallowed up" by the shock.

Now, the shock will have the rarefaction fan to its left: $u_L(x,t) = \frac{x}{t}$



$$u_L(s(t), t) = \frac{s(t)}{t} \quad u_R(s(t), t) = 0 \Rightarrow [[u]] = \frac{s(t)}{t}$$

$$F(u_L(s(t), t)) = \frac{1}{2} \left(\frac{s(t)}{t} \right)^2 \quad F(u_R(s(t), t)) = 0 \Rightarrow [[F(u)]] = \frac{1}{2} \left(\frac{s(t)}{t} \right)^2$$

Rankine-Hugoniot \Rightarrow

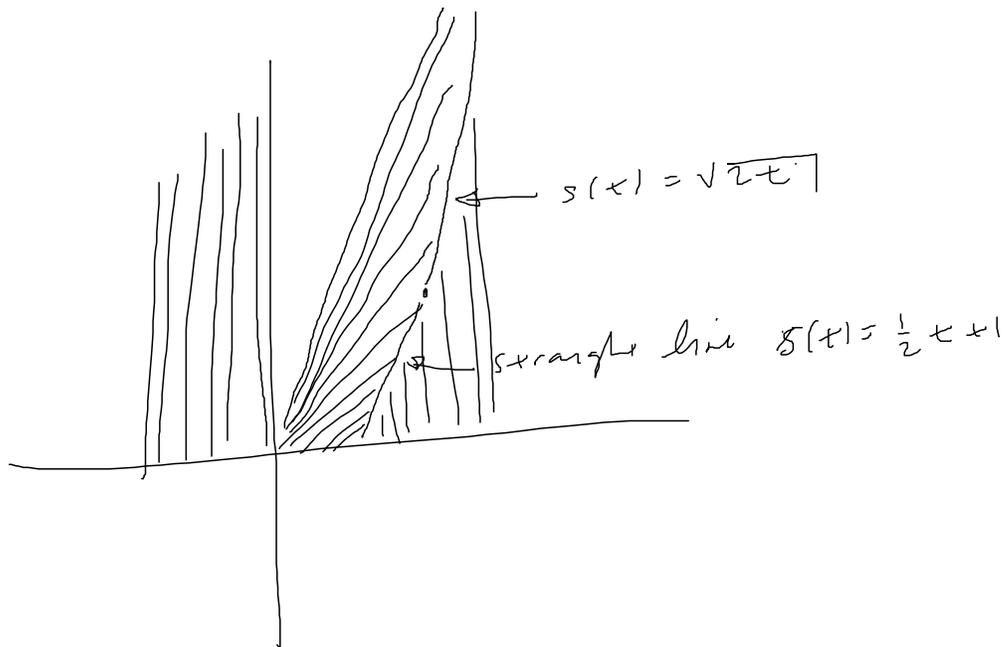
$$\frac{1}{2} \left(\frac{s(t)}{t} \right)^2 = \dot{s} \left(\frac{s(t)}{t} \right) \Rightarrow \dot{s} = \frac{1}{2} \frac{s(t)}{t}$$

Since $s(2) = 2 \Rightarrow s(t) = \sqrt{2t}$

entropy condition satisfied?

$$\text{Yes since } u_L(s(t), t) = \frac{\sqrt{2t}}{t} > \dot{s}(t) = \frac{1}{\sqrt{2t}} > u_R(s(t), t) = 0$$

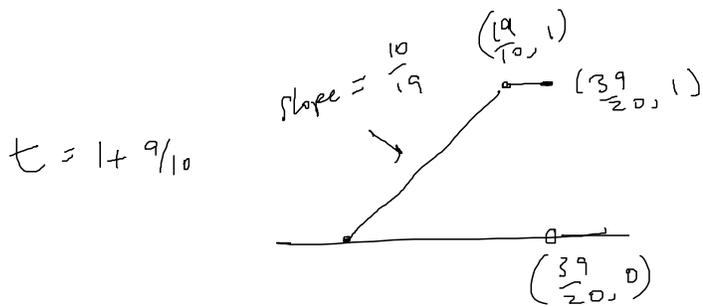
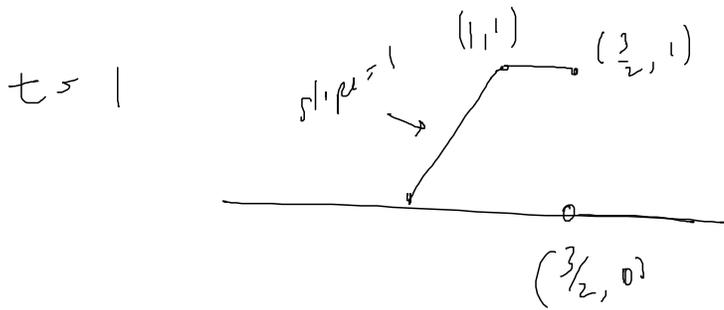
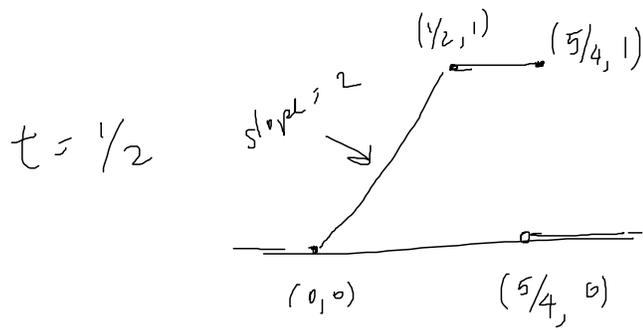
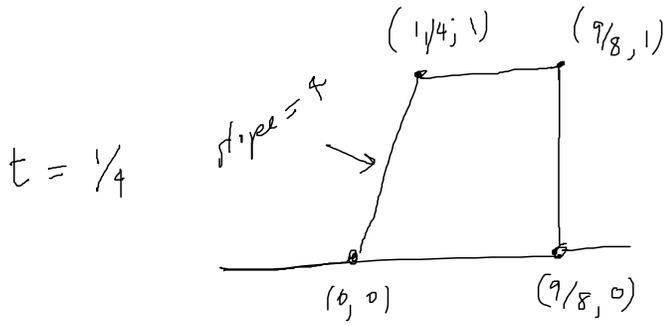
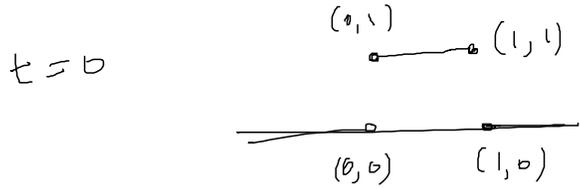
so the shock is as follows:



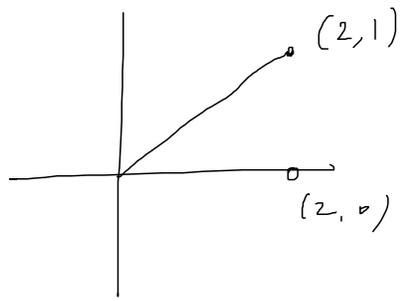
Solution

$$u(x, t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x < t \\ 1 & t < x < 1 + \frac{t}{2} \\ 0 & x > 1 + \frac{t}{2} \end{cases} \quad \text{if } t \leq 2$$

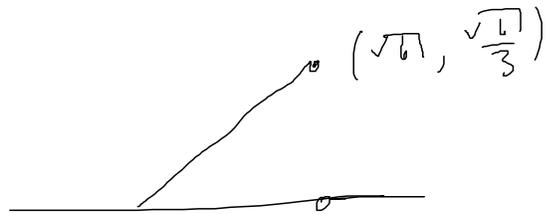
$$u(x, t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x < \sqrt{2t} \\ 0 & x > \sqrt{2t} \end{cases} \quad \text{if } t > 2$$



t = 2

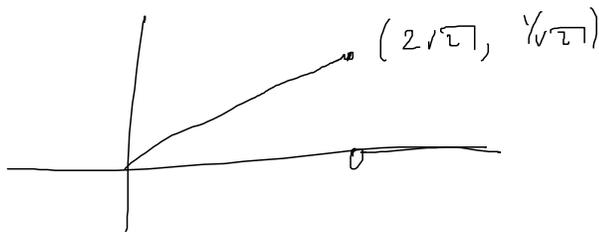


t = 3



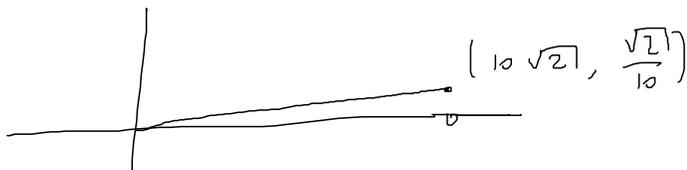
note $\frac{\sqrt{6}}{3} \approx .82$

t = 4



note: $\frac{1}{\sqrt{2}} \approx .71$

t = 100



note: $\frac{\sqrt{2}}{10} \approx .14$