

Special cases

F linear.

$$F(Du, u, x) = \vec{b}(x) \cdot Du(x) + c(x)u(x) = 0 \quad x \in \bar{\Omega}$$

Given a domain $\bar{\Omega}$, the non-characteristic assumption can be written in terms of the normal to the boundary:

$$D_p F(p^*, z^*, x^*) \cdot \nu(x_*) \neq 0$$

(Why? H.W.) For the above PDE, this becomes

$$\vec{b}(\vec{x}) \cdot \nu(x_*) \neq 0$$

It doesn't depend on u or Du at the boundary.

$$F(p, z, x) = \vec{b}(x) \cdot p + c(x)z$$

$$F_p = \vec{b}(x) \quad F_z = c(x) \quad F_x = (\mathcal{D}\vec{b})\vec{p} + z\mathcal{D}c \quad \text{where } (\mathcal{D}\vec{b})_{ij} = \frac{\partial \vec{b}_i}{\partial x_j}$$

$$\frac{d\vec{p}}{ds} = -\mathcal{D}\vec{b}(\vec{x}(s))\vec{p}(s) - z(s)\mathcal{D}c(\vec{x}(s)) - c(\vec{x}(s))\vec{p}(s)$$

$$\frac{dz}{ds} = \vec{b}(x) \cdot \vec{p}(s) = -c(x)z(s)$$

$$\frac{d\vec{x}}{ds}, \vec{b}(x(s))$$

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Note that we don't need the

$$\frac{d\vec{x}}{ds} = \vec{b}(\vec{x}(s))$$

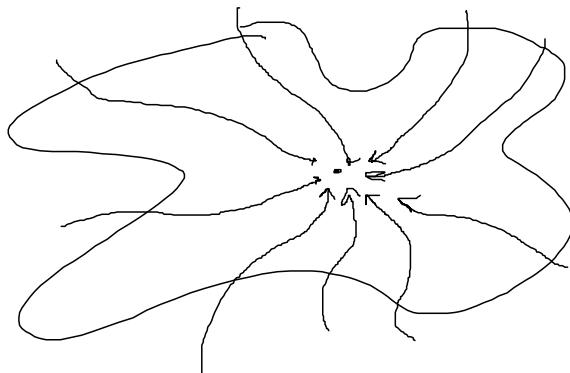
$$\frac{d\vec{x}}{ds} = \vec{b}(\vec{x}(s)) \quad \text{and} \quad \frac{d\vec{z}}{ds} = -c(\vec{x}(s)) \vec{z}(s)$$

\Downarrow

$$\Rightarrow \vec{z}(s) = \vec{z}(0) e^{-\int_0^s c(\vec{x}(\tau)) d\tau}$$

just need to understand
the vector field given by \vec{b} .

ex: $\vec{b} = \vec{0}$ at an attracting point in \mathbb{T}^2 .



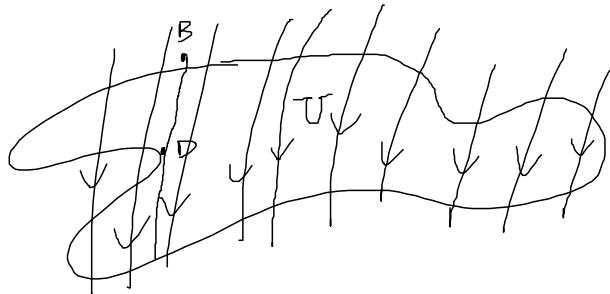
all trajectories tend towards $x_\infty \in \mathbb{T}^2$.

then $\vec{z}(x_0; s) = ?$ depending on x_0 and on $\vec{g}(x_0)$
you could have different
limits. \Rightarrow u may not be continuous at x_∞ !

(For a specific example, consider the case where $c(x) \equiv 0$.
you're guaranteed u will be discontinuous at x_∞ .

(3)

ex: the x-characteristics flow through Γ :



In this case, you would only specify the data g on Γ : $\{x_0 \in \partial\Gamma \mid b(x_0) \cdot \nu(x_0) > 0\} \subset \partial\Gamma$

Note: $D \notin \Gamma$ and so you're not specifying $g(D)$. So it's okay if the value $g(B)$ is determined or at the point D .

ex: scalar conservation laws

$$\begin{aligned} G(Du, u_t, u, x, t) := u_t + \operatorname{div} F(u) &= 0 \\ &= u_t + F'(u) \cdot Du \\ &\quad \text{in } \Gamma = \mathbb{R}^n \times (0, \infty) \end{aligned}$$

subject to initial data $u = g$ on $\Gamma = \mathbb{R}^n \times \{t=0\}$

$$F: \mathbb{R} \rightarrow \mathbb{R}^n$$

$$\text{let } x_{n+1} := t$$

$$\text{then } G(\vec{p}, z, \vec{x}) = p_{n+1} + F'(z) \cdot \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

$$G_p = \begin{pmatrix} F'(z) \\ 1 \end{pmatrix} \quad G_z = F''(z) \cdot \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \quad G_x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ D \end{pmatrix}$$

(7)

the non-characteristic condition is definitely satisfied:

$$G_p(p^*, z^*, x^*) \cdot v(x_*) = 1 \neq 0 \quad \checkmark$$

The characteristic ODE are

$$\frac{d\vec{p}}{ds} = \vec{0} - \left(F''(z) \cdot \begin{pmatrix} p_1 \\ \vdots \\ p_{n+1} \end{pmatrix} \right) \begin{pmatrix} p_1 \\ \vdots \\ p_{n+1} \end{pmatrix}$$

$$\frac{dz}{ds} = \begin{pmatrix} F'(z) \\ 1 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ \vdots \\ p_{n+1} \end{pmatrix} = 0$$

$$\frac{d\vec{x}}{ds} = \begin{pmatrix} F'(z) \\ 1 \end{pmatrix}$$

\Rightarrow don't need the $\frac{d\vec{p}}{ds}$ ODE. $z(\vec{x}(s)) = z(\vec{x}(0))$

just need to understand the $\frac{d\vec{x}}{ds}$ ODE.

$$\frac{dx_1}{ds} = F_1'(z(s))$$

$$\frac{dx_2}{ds} = F_2'(z(s))$$

:

$$\frac{dx_n}{ds} = F_n'(z(s))$$

$$\frac{dx_{n+1}}{ds} = 1$$

Note that since $z(s) = z(0)$ the x-characteristics have fixed slope. They're straight lines and z is constant along these lines.

Note: the $2n+3$ ODE are autonomous and so we know that all curves in $\mathbb{R}^{2n+3}, (\vec{p}(s), z(s), \vec{x}(s))$ cannot cross. However,

The x -characteristics, when viewed in \mathbb{R}^{n+1} , may cross. (The projections of non-crossing curves may then selves cross.)

If the x -characteristics cross then there's a problem at the meeting point: the advected values of the initial data can disagree. You will see in your HW assignment that when this happens, the solution u loses smoothness.

§ 3.4 Introduction to conservation laws.

We now focus on scalar conservation laws in one space dimension:

$$\begin{cases} u_t + (F(u))_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t=0\} \end{cases}$$

by the method of characteristics, we find that you're not guaranteed that a smooth solution exists for all time. So we won't try to define weak solutions.

Q1: what do we do w/ the PDE if we don't have u_t and Du ?

Pretend u is a smooth solution of $u_t + (F(u))_x = 0$.

Let $v : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ be any smooth, compactly supported function.

(5)

Multiply the PDE by v and integrate by parts:

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty [u_t + (F(u))_x] v \, dx dt \\ &= \iint u_t v + \iint (F(u))_x v \\ &= - \iint u v_t - \int_{-\infty}^\infty u(x, 0) v(x, 0) dx \\ &\quad - \iint F(u) v_x \, dx dt \end{aligned}$$

$$\begin{aligned} \Rightarrow 0 &= \int_0^\infty \int_{-\infty}^\infty u(x, t) v_t(x, t) + F(u(x, t)) v_x(x, t) \, dx dt \\ &\quad + \int_{-\infty}^\infty g(x) v(x, 0) \, dx \end{aligned}$$

defn: We say that $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is an integral solution of $u_t + (F(u))_x = 0$ on $\mathbb{R} \times (0, \infty)$ with $v \in C_c^\infty(\mathbb{R} \times \{t=0\})$ if (5) holds for all test functions v that are smooth and have compact support.

Q: what can we conclude about a solution u when all we know about it is that it satisfies (5)?