

PDTs Dec 8, 2004

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In general, we want to solve

$$\begin{cases} Lu = f & \text{in } \bar{\Omega} \\ u = g & \text{on } \partial\Omega \end{cases}$$

If  $v \in C_0^\infty(\bar{\Omega})$  then after integration by parts, we would have

$$B[u, v] = \int_{\Omega} f u v \, dx = (f, v) \quad \text{inner product in } L^2(\Omega)$$

$$\text{where } B[u, v] = \int_{\Omega} Dv^T A(x) Du + (b(x) \cdot Du)v + cuv \, dx$$

Taking the closure of  $C_0^\infty(\bar{\Omega})$  with the  $H^1$  norm, we want

$$B[u, v] = (f, v) \quad \text{for all } v \in H_0^1(\Omega)$$

But what about the boundary data?

If  $u \in H^1(\bar{\Omega})$  then  $u$  might not be continuous up to the boundary! And if  $\partial\Omega \in C^1$  then  $\partial\Omega$  has measure zero, so what does it mean to say that an  $H^1$  function equals  $g$  on a set of measure zero?

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What we say is "n̄g on  $\partial\mathcal{V}$  in the trace sense".

Theorem: (Gagliardo, 1957) Let  $\mathcal{V}$  be open and bounded in  $\mathbb{R}^n$  with a Lipschitz boundary.

Consider the trace operator defined for functions that are Lipschitz continuous on the closure of  $\mathcal{V}$ .

There is a unique continuous extension of the trace operator:  $T : H^1(\mathcal{V}) \rightarrow H^{1/2}(\partial\mathcal{V})$ .

The trace operator is onto  $H^{1/2}(\partial\mathcal{V})$  and  $Tu = 0 \Leftrightarrow u \in H_0^1(\mathcal{V})$ .

If you can't read Italian ☺ a proof of a more elaborate version of the theorem can be found in Grisvard's "Elliptic problems in Nonsmooth Domains".

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What does this mean for us?

Ans: Consider

$$\begin{cases} u = f & \text{on } \bar{\Gamma} \\ u = g & \text{on } \partial\Gamma \end{cases}$$

where  $f \in L^2(\bar{\Gamma})$  and  $g \in H^{\frac{1}{2}}(\partial\Gamma)$ . Then we seek a solution  $u \in H^1(\bar{\Gamma})$  such that

$$B[u, v] = (f, v) \quad \text{L}^2 \text{ inner product}$$

for all  $v \in H_0^1(\bar{\Gamma})$  and such that

$$Tu = g \quad (\text{i.e. } u = g \text{ in the trace sense})$$

We study this problem through an affiliated problem as follows.

Since  $T: H^1(\bar{\Gamma}) \rightarrow H^{\frac{1}{2}}(\partial\Gamma)$  is onto,  $\exists$

$w \in H^1(\bar{\Gamma})$  with  $Tw = g$ .

$$w \in H^1(\bar{\Gamma}) \Rightarrow Lw \in (H^1(\bar{\Gamma}))^*$$

$$\text{and } H_0^1(\bar{\Gamma}) \subseteq H^1(\bar{\Gamma}) \Rightarrow (H^1(\bar{\Gamma}))^* \subseteq (H_0^1(\bar{\Gamma}))^*$$

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$$\Rightarrow Lw \in (H_0^1(\Omega))^*$$

It then follows that  $f - Lw \in (H_0^1(\Omega))^*$

$$\text{Let } f^* := f - Lw$$

We seek a solution of

$$\begin{cases} Lu = f^* & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

If we can find  $\tilde{u}$  such that  $\tilde{u} \in H_0^1(\Omega)$

$$B[\tilde{u}, v] = \langle f^*, v \rangle \quad \forall v \in H_0^1(\Omega)$$

then our final solution is

$$\tilde{u} + w.$$

$$\text{Why? } T(\tilde{u} + w) = T\tilde{u} + Tw = Tw = f$$

and

$$B[\tilde{u} + w, v] = B[\tilde{u}, v] + B[w, v]$$

$$= \langle f^*, v \rangle + \langle Lw, v \rangle$$

$$= \langle f - Lw, v \rangle + \langle Lw, v \rangle$$

$$= \langle f, v \rangle - \langle Lw, v \rangle + \langle Lw, v \rangle = \langle f, v \rangle \quad \checkmark$$

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This shows that if  $g \in H^{\frac{1}{2}}(\partial\Omega)$  and  $f \in L^2(\Omega)$  then to find a weak solution of

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

it suffices to find a weak solution of

$$\begin{cases} Lu = f^* & \text{in } \Omega \\ u \in H_0^1(\Omega) \end{cases}$$

where  $f^* \in (H_0^1(\Omega))^*$

Theorem: Assume  $\Omega$  open + bounded and connected.

$\partial\Omega \in C^1$  and

$$Lu = -\operatorname{div}(ADu)$$

is uniformly elliptic

Assume  $A_{ij} \in L^\infty(\Omega)$  for all  $i, j$ .

Then  $\exists ! u \in H_0^1(\Omega)$  such that

$$B[u, v] = \langle f^*, v \rangle$$

$v \in H_0^1(\Omega)$

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Proof: By the Lax-Milgram theorem, it suffices to find  $\alpha, \beta > 0$  such that

$$|B[u, v]| \leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \quad \forall u, v$$

and

$$\beta \|u\|_{H_0^1(\Omega)}^2 \leq B[u, u] \quad \forall u \in H_0^1(\Omega)$$

$$|B[u, v]| = \left| \int_{\Omega} Dv^T A Du \, dx \right|$$

We assumed that  $A_{ij} \in L^\infty(\Omega) \quad \forall i, j$ .

$$\text{Let } C = \max \|A_{ij}\|_{L^\infty(\Omega)}.$$

$$\text{Then } \left| \int_{\Omega} Dv^T A Du \, dx \right| \leq \int_{\Omega} \left| \sum_{i,j} v_{x_i} A_{ij} u_{x_j} \right| \, dx$$

$$\leq \int_{\Omega} \sum_{i,j} |v_{x_i}| |A_{ij}| |u_{x_j}| \, dx$$

$$\leq C \int_{\Omega} \sum_{i,j} |v_{x_i}| |u_{x_j}| \, dx$$

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$$\leq C \int_{\Omega} \sum_{i,j} |Dv| |Du| dx$$

$$|Dv| = \sqrt{\sum_i (v_{x_i})^2}$$

$$= Cn^2 \int_{\Omega} |Dv| |Du| dx$$

$$\leq Cn^2 \sqrt{\int_{\Omega} |Dv|^2 dx} \sqrt{\int_{\Omega} |Du|^2 dx}$$

$$\leq Cn^2 \sqrt{\int_{\Omega} |v|^2 + |Dv|^2 dx} \sqrt{\int_{\Omega} |u|^2 + |Du|^2 dx}$$

Taking  $\alpha = Cn^2$  we have shown

$$|B[u, v]| \leq \alpha \|v\|_{H_0^1(\Omega)} \|u\|_{H_0^1(\Omega)}.$$

Now for the lower bound!

recall that  $Lu$  is assumed to be uniformly elliptic  $\exists \theta > 0$  such that

$$\Rightarrow \theta |\zeta|^2 \leq \zeta^T A(x) \zeta \quad \forall \zeta \in \mathbb{R}^n$$

then

$$B[u, u] = \int_{\Omega} Du^T A Du \, dx$$

$$\Theta \int_{\Omega} |Du|^2 \, dx \leq \int_{\Omega} Du^T A Du \, dx.$$

now we want  $\|u\|_{H_0^1(\Omega)}^2$  on the left hand side. That is, we need  $\beta > 0$  so that

$$\beta \int_{\Omega} |u|^2 + |Du|^2 \, dx \leq \Theta \int_{\Omega} |Du|^2 \, dx$$

since then we'll have

$$\beta \int_{\Omega} |u|^2 + |Du|^2 \, dx = \beta \|u\|_{H_0^1(\Omega)}^2 \leq B[u, u]$$

and we're done by Lax-Milgram theorem.

Lemma: (Poincaré Inequality)

Assume  $\Omega$  is open & bounded & connected and  $\partial\Omega \in C^1$  then  $\exists c$  that depends only on  $\Omega$  & the dimension of  $\mathbb{R}^n$  such that if  $u \in H_0^1(\Omega)$  then

$$\|u\|_{L^2} \leq c \|Du\|_{L^2}$$

(See page 275 in Evans)

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$$\text{Take } \beta = \frac{\theta}{c+1}$$

$$\begin{aligned} \text{then } \beta \|u\|_{L^2(\Omega)}^2 + \beta \|Du\|_{L^2}^2 \\ \leq c^2 \beta \|Du\|_{L^2}^2 + \beta \|Du\|_{L^2}^2 \\ = \beta (c^2 + 1) \|Du\|_{L^2}^2 = \theta \|Du\|_{L^2(\Omega)}^2 \end{aligned}$$

$$\Rightarrow \beta \|u\|_{H_0^1(\Omega)}^2 \leq \theta \|Du\|_{L^2(\Omega)}^2 \leq \beta [u, u] \text{ and}$$

We're done by the Lax-Milgram theorem. //

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What is this Poincaré inequality?

Here's a proof if  $V = (a, b) \subseteq \mathbb{R}$

$C_0^\infty(V)$  is dense in  $H_0^1(V)$  so we prove

Poincaré's inequality for  $u \in C_0^\infty(V)$ .

$$u(x) - u(a) = \int_a^x D_u(y) dy$$

$$\Rightarrow u(x) = \int_a^x D_u(y) dy$$

$$\Rightarrow \int_a^b |u(x)|^2 dx = \int_a^b \left[ \int_a^x D_u(y) dy \right]^2 dx$$

$$\leq \int_a^b \left[ \int_a^x |D_u(y)| dy \right]^2 dx$$

recall:  $\int_a^x |D_u(y)| dy \leq \sqrt{\int_a^x |D_u(y)|^2 dy} \sqrt{x-a}$

$$\leq \sqrt{\int_a^b |D_u(y)|^2 dy} \sqrt{b-a}$$

$$\Rightarrow \int_a^b |u(x)|^2 dx \leq \int_a^b b-a \int_a^b |D_u(y)|^2 dy dx$$

$$\Rightarrow \int_a^b |u(x)|^2 dx \leq (b-a)^2 \int_a^b |Du(y)|^2 dy.$$

$$\Rightarrow \|u\|_{L^2} \leq (b-a) \|Du\|_{L^2}$$

↗ constant that depends on  
 $\mathcal{J}$  and dimension.

Another way to do this is via Fourier Series.

Alternate version of Poincaré theorem:

let  $\bar{u}$  = average value of  $u$  in  $\mathcal{J}$

$$= \frac{1}{\text{meas } \mathcal{J}} \int_{\mathcal{J}} u(y) dy.$$

then  $\|u - \bar{u}\|_{L^2(\mathcal{J})} \leq C \|Du\|_{L^2(\mathcal{J})}$ .

Assume  $\mathcal{J} = (-b, b) \Rightarrow$

$$u - \bar{u} = \sum_{k=-\infty, k \neq 0}^{\infty} u_k e^{ikx \frac{\pi}{b}}$$

$$Du = \sum_{n=-\infty, n \neq 0}^{\infty} ik \frac{\pi}{b} u_n e^{ikx \frac{\pi}{b}}$$

$$\Rightarrow \|u - \bar{u}\|_{L^2}^2 = \sum_{\substack{k=1 \\ k \neq 0}}^{\infty} |u_k|^2$$

$$\|Du\|_{L^2}^2 = \sum_{-\infty}^{\infty} k^2 \left(\frac{\pi}{b}\right)^2 |u_k|^2$$

$$\Rightarrow \|u - \bar{u}\|_{L^2}^2 = \left(\frac{b}{\pi}\right)^2 \sum_{\substack{-\infty \\ k \neq 0}}^{\infty} \left(\frac{\pi}{b}\right)^2 k^2 |u_k|^2$$

$$= \left(\frac{b}{\pi}\right)^2 \|Du\|_{L^2}^2$$

$$\Rightarrow \|u - \bar{u}\|_{L^2} \leq \left(\frac{b}{\pi}\right) \|Du\|_{L^2}$$

Note:  $\left(\frac{\pi}{b}\right)^2$  is the smallest eigenvalue of  $-\Delta$  on  $[-b, b]$ . This is related to the Poincaré inequality since  $-\Delta u = \lambda u$

$$\Rightarrow \int (-\Delta u)(u) = \int \lambda u u$$

$$\Rightarrow \int |Du|^2 = \lambda \int |u|^2 dx \Rightarrow \|u\| \leq \frac{1}{\sqrt{\lambda}} \|Du\|$$

holds for any eigenfunction.  $\Rightarrow C \geq \max\left\{\frac{1}{\sqrt{\lambda}}\right\} = \frac{b}{\pi}$

Okay, so with the aid of Poincaré's inequality, we prove a  $\exists!$  weak solution

$$\begin{cases} -\operatorname{div}(A \nabla u) = f^* & \text{in } \mathcal{V} \\ u \in H_1^0(\mathcal{V}) \end{cases}$$

We didn't really need  $\mathcal{V}$  to be connected.

If  $\mathcal{V}$  were simply bounded, we could

use Poincaré on each connected component of  $\mathcal{V}$ :

$$\begin{aligned} \int_{\mathcal{V}} |u|^2 &\leq \sum_i \int_{\mathcal{V}_i} |u|^2 dx \leq \sum_i C_i^{-2} \int_{\mathcal{V}_i} |\nabla u|^2 dx \\ &\leq C^2 \sum_i \int_{\mathcal{V}_i} |\nabla u|^2 dx = C^2 \int_{\mathcal{V}} |\nabla u|^2 dx. \end{aligned}$$

and this would then give us our lower bound.

What about the general case of

$$Lu = -\operatorname{div}(A(x)Du) + \vec{b}(x) \cdot Du + c(x)u$$

?

First, let's see how much of the estimates we can get for the Lax-Milgram theorem.

Theorem: Assume  $\Omega$  is open, bounded & connected and  $\partial\Omega$  is  $C^1$ . Assume  $A_{ij}, b_i, c \in L^\infty(\Omega)$  for all  $i, j$ . Assume  $A$  is uniformly elliptic

Let

$$B[u, v] := \int_{\Omega} Dv^T A D u + (\vec{b} \cdot Du) v + c u v \, dx$$

then  $\exists \alpha, \beta$ , and  $\gamma > 0$  such that

$$|B[u, v]| \leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}$$

$$\beta \|u\|_{H_0^1(\Omega)}^2 \leq B[u, u] + \gamma \|u\|_{L^2(\Omega)}^2$$

for all  $u, v \in H_0^1(\Omega)$ .

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Proof: The proof will be very similar to before.

Step 1:

$$\begin{aligned}
 |\mathcal{B}(u, v)| &\leq \int_V |Dv^T A Du| + |\vec{b} \cdot Du| |v| + |c| |u| |v| \, dx \\
 &\leq \sum_{ij} |A_{ij}|_{L^\infty} \int_V |Dv| |Du| + \sum_i |\vec{b}_i|_{L^\infty} \int_V |Du| |v| \\
 &\quad + |c|_{L^\infty} \int_V |u| |v| \, dx \\
 &\leq C \left[ \sqrt{\int |Du|^2} \right] \sqrt{\int |Dv|^2} + \sqrt{\int |Du|^2} \left[ \sqrt{\int |v|^2} \right] \\
 &\quad + \sqrt{\int |u|^2} \left[ \sqrt{\int |v|^2} \right] \\
 &\leq C \|u\|_{H_0^1(V)} \|v\|_{H_0^1(V)}
 \end{aligned}$$

Step 2:

$$\begin{aligned}
 \Theta \int |Du|^2 &\leq \int (Du)^T A Du \\
 &= \mathcal{B}[u, u] - \int (\vec{b} \cdot Du) u - \int c u^2 \, dx
 \end{aligned}$$

$$\Rightarrow \theta \int |Du|^2 \leq B[u, u] + \sum_i \|b_i\|_{L^\infty} \int |u| |Du| + \|c\|_{L^\infty} \int |u|^2 dx$$

$$\left| \int |u| |Du| dx \right| \leq \sqrt{\int |u|^2 dx} \sqrt{\int |Du|^2 dx}$$

$$\leq \varepsilon \int |Du|^2 + \frac{1}{4\varepsilon} \int |u|^2$$

$$\text{Since } ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

$$\Rightarrow (\sqrt{2\varepsilon}a)(\frac{b}{\sqrt{2\varepsilon}}) \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$$

idea: by taking  $\varepsilon \ll 1$ , we can choose  $\varepsilon$  so that

$$\sum_i \|b_i\|_{L^\infty} \int |u| |Du| \leq \frac{\theta}{2} \int |Du|^2 + C \int |u|^2$$

here  $C$  depends on  $\varepsilon$  and  $\varepsilon$  depends on  $\|b_i\|_{L^\infty}$  and  $\theta$ :  $\sum_i \|b_i\|_{L^\infty} \leq \frac{\theta}{2}$

thus this allows us to do the following

$$\theta \int |Du|^2 \leq B[u, u] + \frac{\theta}{2} \int |Du|^2 + C \int |u|^2 + \|c\|_{L^\infty} \int |u|^2$$

$\Rightarrow$  subtract  $\frac{\theta}{2} \int |Du|^2$  from both sides

$$\frac{\theta}{2} \int |Du|^2 \leq B[u, u] + \gamma \int |u|^2$$

where  $\gamma$  is a new, larger constant.

that now depends on  $\theta, \|b\|_{L^\infty}, \|c\|_{L^\infty}$

Okay, we'd be done if the LHS were

$$\beta \|u\|_{H_0^1}^2 = \int |u|^2 + |Du|^2 dx$$

but we only have  $\int |Du|^2$ . We now use Poincaré's inequality as before and find

$$\|u\|_{L^2} \leq C \|Du\|_{L^2}$$

$\Rightarrow$  if  $\beta := \frac{\theta/2}{C^2 + 1}$  then

$$\beta \|u\|_{H_0^1(\Gamma)}^2 \leq \frac{\theta}{2} \|Du\|_{L^2(\Gamma)}^2 \leq B[u, u] + \gamma \|u\|_{L^2}^2$$

as desired. //

Lemma: Assume  $\Omega$  is open, bounded, connected and  $\partial\Omega$  is  $C^1$ . Assume  $A_{ij}, b_i, c \in L^\infty(\Omega)$ .

Assume  $L$  is uniformly elliptic

Then  $\exists \gamma > 0$  such that if

$$\mu \geq \gamma$$

and  $f^* \in (H_0^1(\Omega))^*$  then  $\exists$  weak solution  $u$  of

$$\begin{cases} Lu + mu = f^* & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

That is  $\exists u \in H_0^1(\Omega)$  such that

$$B_\mu[u, v] = \langle f^*, v \rangle \quad \forall v \in H_0^1(\Omega)$$

where  $B_\mu[u, v] := B[u, v] + \mu \int_{\Omega} uv dx$

and  $B[u, v] = \int_{\Omega} Dv^T A Du + (\vec{b} \cdot Du) v + cuv dx$

Proof: By previous result,  $\exists \alpha, \beta, \gamma$

$$|B[u, v]| \leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}$$

$$\beta \|u\|_{H_0^1(\Omega)}^2 \leq B[u, u] + \gamma \|u\|_{L^2(\Omega)}^2$$

Since  $B_m[u, v] = B[u, v] + m \int_{\Omega} uv dx$

If  $M \geq \gamma$  then  $\exists \alpha_m$  and  $\beta_m > 0$  such that

$$|B_m[u, v]| \leq \alpha_m \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}$$

and

$$\beta_m \|u\|_{H_0^1(\Omega)}^2 \leq B_m[u, u]$$

for all  $u, v \in H_0^1(\Omega)$ . Now we're done  
by the Lax-Milgram theorem. //

Note: the corollary proves that  $\exists \gamma \exists \gamma$  if  $M \geq \gamma$  then

$$L_m := L + mI : H_0^1(\Omega) \rightarrow (H_0^1(\Omega))^*$$

is an isomorphism.

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Q: What about solving  $Lu = f^*$  in general?

See Thm 4 on page 303. Need to use  
Fredholm alternative.