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2<sup>nd</sup> order elliptic equations.

Let  $\Omega \subseteq \mathbb{R}^n$  be open & bounded we will consider the problem

$$\begin{cases} Lu = f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

$u: \bar{\Omega} \rightarrow \mathbb{R}$  is the unknown,  $f: \bar{\Omega} \rightarrow \mathbb{R}$  is given. The linear operator  $L$  will be assumed to take one of the forms

$$(1) \quad Lu = - \sum_{i=1}^n \sum_{j=1}^n (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^n b_i(x)u_{x_i} + c(x)u(x)$$

$$(2) \quad Lu(x) = - \sum_{i=1}^n \sum_{j=1}^n a^{ij}(x)u_{x_i}u_{x_j} + \sum_{i=1}^n b_i(x)u_{x_i} + c(x)u(x)$$

In (1) the second-order term is in divergence form

$$Lu = -\operatorname{div}(A(x)Du) + b(x) \cdot Du + c(x)u$$

In the first case, we say the PDE

$$Lu = f$$

is "in divergence form".

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Note if  $a^{ij}(x) \in C^1(\bar{\Omega})$  for all  $i, j$  then

an operator in divergence form can be written in non divergence form & vice versa.

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The divergence form will be best for energy methods and the non divergence form will be best for maximum principle techniques.

We assume  $A(x)$  is symmetric  $A(x)^T = A(x)$

defn: the operator  $L$  is uniformly elliptic

if  $\exists \theta > 0$  ( $\theta$  a constant) such that

$$\{^TA(x)\} \geq \theta |\zeta|^2 \quad \forall \zeta \in \mathbb{R}^n \quad \forall x \in \bar{\Omega}$$

i.e.  $\langle u, v \rangle := \langle A(x)u, v \rangle$  is a legit inner product.

the matrix  $A(x)$  is positive definite with smallest eigenvalue greater than or equal to 0.

Note:  $A = I$ ,  $\vec{b} = \vec{0}$ ,  $c(x) = 0$

$$\Rightarrow L = -\Delta$$

Physically, the

$\nabla \cdot (A(x) \nabla u)$  reflects diffusion

$\vec{b}(x) \cdot \nabla u$  reflects advection

$c(x)u$  reflects creation/depletion

We will consider

$$a^{ij}, b^i, c \in L^\infty(\bar{\Omega})$$

$$f \in L^2(\Omega)$$

and will seek a weak solution

$$f \quad \begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

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If the PDE is in divergence form & if we had a smooth solution and smooth  $\partial\Omega$  and  $v \in C_c^\infty(\bar{\Omega})$  ( $i.e.$   $\text{supp}(v) \subseteq \bar{\Omega}$ ) then

$$\begin{aligned} \int_{\Omega} (Lu)v \, dx &= \int_{\Omega} -\operatorname{div}(A(x)Du)v + (\vec{b} \cdot Du)v + c(x)uv \, dx \\ &= \int_{\Omega} A(x)Du \cdot Dv + v \vec{b} \cdot Du + cuv \, dx = \int_{\Omega} fv \, dx \end{aligned}$$

$$B[u, v] := \int_{\Omega} A(x)Du \cdot Dv + v \vec{b} \cdot Du + cuv \, dx$$

$\Omega$  is a bi-linear form

Introduce a function space  $H_0^1(\Omega)$  as follows.

$$H_0^1(\Omega) = \overline{C_c^\infty(\bar{\Omega})}$$

where the norm is taken w.r.t the

norm  $\|u\| = \sqrt{\int |\nabla u|^2 + |Du|^2 \, dx}$

Note that  $H_0^1(\Omega)$  has the Dirichlet boundary conditions built in.

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Defn:  $u \in H_0^1(\Omega)$  is a weak solution of

$$\begin{cases} -\operatorname{div}(A(x)Du) + b(x) \cdot Du + c(x)u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

if  $B[u, v] = \langle f^*, v \rangle$  for all  $v \in H_0^1(\Omega)$

where  $f^*$  is in  $(H_0^1(\Omega))^*$

We use the Lax-Milgram theorem to prove  $\exists$  a weak solution.

Thm: (Lax-Milgram). Assume  $H$  is a Hilbert space with inner product  $(\cdot, \cdot)$ . Assume

$$B : H \times H \rightarrow \mathbb{R}$$

is a bilinear mapping for which  $\exists$  constants  $\alpha, \beta > 0$  such that

$$|B[u, v]| \leq \alpha \|u\| \|v\| \quad \forall u, v \in H$$

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and

$$S\|u\|^2 \leq B[u, u] \quad \forall u \in H$$

Finally, let  $f$  be a bounded linear functional on  $H$ . Then  $\exists ! u \in H$  such that

$$B[u, v] = \langle f, v \rangle \quad \forall v \in H^*, (v)$$


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To prove this, we'll need the Riesz Representation theorem:

$H^* = \{ \text{the space of bounded linear functionals on } H \}$

$$= \{ u^* : H \rightarrow \mathbb{R} \quad \text{a bounded linear map} \}$$

$$\text{If } w \in H \text{ then } u^*(w) =: \langle u^*, w \rangle$$

Riesz Representation theorem: For each  $u^* \in H^*$   $\exists ! u \in H$  such that

$$\langle u^*, v \rangle = (u, v) \quad \forall v \in H$$

and  $u^* \mapsto u$  is a linear isomorphism of  $H^*$  onto  $H$ .

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## Proof of Lax-Milgram Theorem:

fix  $u \in H$ .

then  $v \mapsto B[u, v]$  is a bounded linear functional. Why?

$$|B[u, v]| \leq (\alpha \|u\|) \|v\|$$

↗ the norm of the functional

Hence by Riesz Representation Theorem  $\exists! w \in H$  such that

$$B[u, v] = (w, v) \quad \forall v \in H.$$

We denote this mapping  $u \mapsto w$  by

$$w = Au \quad \text{i.e.}$$

$$B[u, v] = (Au, v) \quad \forall u, v \in H$$

Claim:  $A: H \rightarrow H$  is a bounded linear operator.

Is it linear?

$$\begin{aligned} (A(\lambda_1 u_1 + \lambda_2 u_2), v) &= B[\lambda_1 u_1 + \lambda_2 u_2, v] \\ &= \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\ &\quad \text{by bilinearity} \\ &\Rightarrow \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v) \end{aligned}$$

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$$= (\lambda_1 A u_1 + \lambda_2 A u_2, v) \quad \forall v.$$

So since

$$(A(\lambda_1 u_1 + \lambda_2 u_2), v) = (\lambda_1 A u_1 + \lambda_2 A u_2, v) \quad \forall v.$$

we have  $A(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 A u_1 + \lambda_2 A u_2$

and  $A$  is linear; as desired.

is  $A$  bounded?

$$\|Au\| \leq c \|u\| \quad \forall u ?$$

know  $\|Au\|^2 = (Au, Au)$

$$= B[u, Au]$$

$$\leq \alpha \|Au\| \|u\|$$

$$\Rightarrow \|Au\|^2 \leq \alpha \|u\| \quad \forall u. \checkmark$$

claim:  $A$  is one-to-one and  $R(A)$

the range of  $A$  is closed in  $H$ .

recall

$$\beta \|u\|^2 \leq B[u, u] = (Au, u) \leq \|Au\| \|u\|$$

$$\Rightarrow \beta \|u\| \leq \|Au\|.$$

thus prove A is l: l since  $Aw \rightarrow w$

Now to show  $R(A)$  is closed. Take a convergent sequence  $\{z_n\} \subset R(A)$ :

$$Au_n \rightarrow v \in H$$

$w \in R(A)$  some  $w \in H$ .

Since  $\{Au_n\}$  is convergent,  $\{u_n\}$  is Cauchy.

$\Rightarrow \{u_n\}$  is Cauchy since  $\beta \|u_n - u_m\| \leq \|Au_n - Au_m\|$

$\Rightarrow u_n \rightarrow w$  some  $w \in H$ . Since  $H$  complete.

And A continuous  $\Rightarrow Au_n \rightarrow Aw$ . By uniqueness of limits,  $Aw = v$ .

claim:  $R(A) \neq H$

Assume not  $R(A)$  is a closed subspace of  $H$ .

Take  $w \in R(A)^\perp$   $w \neq 0$ . Then

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$$(Av, w) = 0 \quad \forall v \in H$$

$$\text{specifically, } (Aw, w) = 0.$$

$$\text{But } 0 < \|w\|^2 \leq B[w, w] = (Aw, w) = 0. \quad \times$$

Okay, we've analyzed the bilinear form  $B[u, v]$

We want to prove  $\exists u \in H$   $B[u, v] = \langle f, v \rangle$   
for all  $v \in H$ . So we turn our gaze to the RHS.

by the Riesz Representation theorem,  $\exists ! w \in H$   
such that  $\langle f, v \rangle = (w, v) \quad \forall v \in H$

Since  $R(A) = H$ ,  $w = Au$  for some  $u \in H$ .

Hence  $B[u, v] = (Au, v) = (w, v) = \langle f, v \rangle \quad \forall v \in H$ .

Now t. show  $u$  is unique. Assume

$$B[u, v] = \langle f, v \rangle = B[\tilde{u}, v] \quad \forall v \in H$$

then  $(Au, v) = (\tilde{u}, v) \quad \forall v \in H$

$$\Rightarrow (Au - \tilde{u}, v) = 0 \quad \forall v \in H$$

$$\Rightarrow Au - \tilde{u} = 0 \Rightarrow u = \tilde{u} \quad \text{since } A \text{ l.i.} \quad //$$