In doing this problem, you will construct a solution of

$$\begin{cases} u_t = u_{xx} & \text{on} \quad \mathbb{R} \times (0, \infty) \\ u(x,t) = 0 & \text{on} \quad \mathbb{R} \times \{t = 0\} \end{cases}$$

that has the following properties:

1. $u(x,0) = 0$ on $\mathbb{R}$ but $u \not\equiv 0$ on $\mathbb{R} \times (0, \infty)$.
2. $u \in C^\infty(\mathbb{R}^2)$
3. For fixed $t$, the function $x \to u(x,t)$ is an entire analytic function on $\mathbb{R}$.
4. The function $t \to u(0,t)$ is not analytic on $\mathbb{R}$.
5. Given $T > 0$, you cannot find $A, a > 0$ such that $|u(x,t)| \leq A \exp(a|x|^2)$ for all $t \in [0,T]$.
6. Finally, you can construct infinitely many of these peculiar beasts, and so you've found infinitely many nonzero solutions of the heat equation, all of which have zero initial data.

**Step 1.** First, you will construct your solution $u(x,t)$ by prescribing data on the $t$-axis: $u(0,t) = g(t)$ and $u_x(0,t) = 0$ for all $t$. Assume $u(x,t)$ has a power series expansion:

$$u(x,t) = \sum_{j=0}^{\infty} g_j(t) x^j.$$

Using the prescribed data, find $g_j(t)$ in terms of $g(t)$.

**Step 2.** Now, fix $\alpha > 0$ and take $g(t)$ as follows:

$$g(t) = \begin{cases} \exp(-t^{-\alpha}) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

Show that there exists $\theta = \theta(\alpha) > 0$ such that for all $t > 0$

$$|g^{(k)}(t)| \leq \frac{k!}{(\theta t)^k} \exp(-\frac{1}{2} t^{-\alpha}).$$

Hint: recall Cauchy's derivative estimate:

$$|g^{(k)}(z)| = \left| \frac{k!}{2\pi i} \int_\Gamma \frac{f(y)}{(y-z)^{1+k}} dy \right| \leq M k! \, r^{-k}$$

which holds if $\Gamma$ is a circle of radius $r$ centered at $z$ and $f$ satisfies the bound $|f(y)| \leq M$ on $\Gamma$. Find bounds for $g^{(k)}(t)$ from Cauchy's derivative estimate applied to $g(z) = \exp(-z^{-\alpha})$ in the complex plane, using as a path of integration a circle $\Gamma$ with centre $t$ and radius $\theta t$, with $\theta$ taken so small that $\mathrm{Re}(x^{-\alpha}) > \frac{1}{2}t^{-p}$ on $\Gamma$.

**Step 3.** Using the Step 2 and the bound $k!/(2k)! \le 1/k!$, show that for any $t$ your power series can be majorized by the power series for

$$U(x,t) = \begin{cases} \exp[\frac{1}{t}\left(\frac{x^2}{\theta} - \frac{1}{2}t^{1-\alpha}\right)] & \text{for } t > 0 \\ 0 & \text{for } t \le 0. \end{cases}$$

Conclude that your power series for $u(x,t)$ converges uniformly to 0 on bounded sets of $\mathbb{R}$ as $t \to 0$.

**Step 4.** Since $U(x,t)$ is bounded uniformly for bounded $x$ and all $t$, show that your power series for $u(x,t)$ converges uniformly in $x$ and $t$ for bounded $x$ and all real $t$. Conclude that $u(x,t)$ is continuous on $\mathbb{R}^2$. If you formally differentiate your power series in $x$, you find a new power series. Show that this power series converges uniformly in $x$ and $t$ for bounded $x$ and all real $t$ and conclude that $u_x(x,t)$ is continuous on $\mathbb{R}^2$. Convince yourself that however many $x$-derivatives you formally take of your power series, you get a power series that converges uniformly in $x$ and $t$ for bounded $x$ and all real $t$, and conclude that $u$ has infinitely many $x$-derivatives on $\mathbb{R}^2$.

**Step 5.** Formally verify that

$$\sum_{k=2}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!}x^{2k-2} = \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)}{(2k)!}x^{2k}$$

and this power series converges uniformly in $x$ and $t$ for bounded $x$ and all real $t$. Conclude that $u_t(x,t)$ is continuous on $\mathbb{R}^2$ and equals $u_{xx}$. Convince yourself that at the formal level the relation $(\partial/\partial t)^k u = (\partial/\partial x)^{2k} u$ holds and that the resulting series converge uniformly in $x$ and $t$ for bounded $x$ and all real $t$. Conclude that $(\partial/\partial t)^k u$ is continuous on $\mathbb{R}^2$. Conclude that $u \in C^\infty(\mathbb{R}^2)$.

**Step 6.** Observe that $u(x,t)$ is an entire analytic function of $x$ for any real $t$ but is not analytic in $t$ since $u(0,t)$ vanishes for $t \le 0$ but not for $t > 0$. Observe that $u$ achieves the initial data, as claimed. Observe that you if you take $\alpha > 1$ then you cannot find $A$ and $a$ so that the bound $|u(x,t)| \le A\exp(a|x|^2)$ holds on $\mathbb{R} \times [0,T]$.