p. 54. #5 (a) $\overset{[M]}{\overset{\shortparallel}{cl(M)}}$ = set of all contact points

$= \{ x \in M \mid$ every nbhd of x contains atleast one point in $M \}$

$\supseteq \{ y \in M \mid$ every nbhd of y contains infinitely many points of $M \}$

$\Rightarrow cl(M)$ is closed.

(b) Suppose A is a closed set and $M \subset A$

let $x \in cl(M) = [M]$

if $x \in M$ then $x \in A$.

if $x \notin M$ and x is a limit point of M
$\Rightarrow x$ is a limit point of A.
$\Rightarrow x \in A$ since A closed.

~~if $x \notin M$ and x is not a limit pt of M.~~
~~$\Rightarrow$ every nbhd of x contains finitely many points of M.~~

if $x \notin M$ and $x \notin A$
$\exists$ an open set $U \subsetneq A^c$ s.t. $x \in U$
$\Rightarrow$ not every open set containing x, contains points of M.
$\Rightarrow x \notin cl(M)$

so, $x \in [M] \Rightarrow x \in A$.

so, $[M] \subset A$

so, $[M]$ is the smallest closed set containing M.

p55. #11. $M_K = \{ f \in C_{[a,b]} \mid |f(t_1) - f(t_2)| \leq K |t_1 - t_2| \quad \forall t_1, t_2 \in [a,b] \}$

$C_{[a,b]} = \{ \text{all continuous functions on } [a,b] \}$

let $U = \{ f \text{ differentiable on } [a,b] \mid |f'(t)| \leq K \}$

(a) Let $f_n \in M_K \quad \forall n$ we need to show $f_n \to f \in M_K$

Suppose $f_n \to f$.

Then
$$|f(t_1) - f(t_2)| = |f(t_1) - f_n(t_1) + f_n(t_1) - f(t_2) + f_n(t_2) - f_n(t_2)|$$

$$\leq |f(t_1) - f_n(t_1)| + |f_n(t_2) - f(t_2)| + |f_n(t_1) - f_n(t_2)|$$

$$= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + K|t_1 - t_2|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \quad \text{if } |t_1 - t_2| < \frac{\varepsilon}{3K} = \delta \text{ and } n \geq N$$

$$= \varepsilon$$

so, $\forall \varepsilon > 0 \ \exists \delta$ s.t. $|f(t_1) - f(t_2)| < \varepsilon$ if $|t_1 - t_2| < \delta$

$\Rightarrow f$ is continuous.

Now $|f(t_1) - f(t_2)| \leq \frac{2\varepsilon}{3} + k|t_1 - t_2|$

Take $\varepsilon \to 0$.

$\Rightarrow |f(t_1) - f(t_2)| \leq k|t_1 - t_2|$

$\Rightarrow f \in M_k$.

$\Rightarrow M_k$ closed.

Claim: $M_k = [U]$

($\Rightarrow$) suppose $f \in M_k$.

$$|f(t) - f(t+h)| \leq k|h|$$

$$\left| \frac{f(t) - f(t+h)}{h} \right| \leq k.$$

take $h \to 0$, if f diff

$$|f'(t)| \leq K.$$

$\Rightarrow f \in U$

if f is not differentiable.

f must be limit point of a sequence of differentiable functions!

Weierstrass approximation thm p186 Gordon, R.A. "Real analysis A first course" $\Big\{$ all continuous function can approximated arbitraily closely by polynomials.

$\Rightarrow f \in [U]$

($\Leftarrow$) suppose $f \in U$.

Mean value thm $\Rightarrow$

$\exists\, c \in (a, b)$ s.t.

$f(b) - f(a) = f'(c)(b-a)$

$\Rightarrow |f(b) - f(a)| \leq |f'(c)||b-a|$

take $b = t_1$, $a = t_2$

since f diff on $[a,b]$

$\Rightarrow f$ diff on $[t_1, t_2]$

$t_1, t_2 \in [a,b]$.

$\Downarrow$

$|f(t_1) - f(t_2)| \leq |f'(\tilde{c})| |t_1 - t_2|$

$|f'(\tilde{c})| \leq K$.

$\Downarrow$

$f \in M_k$.

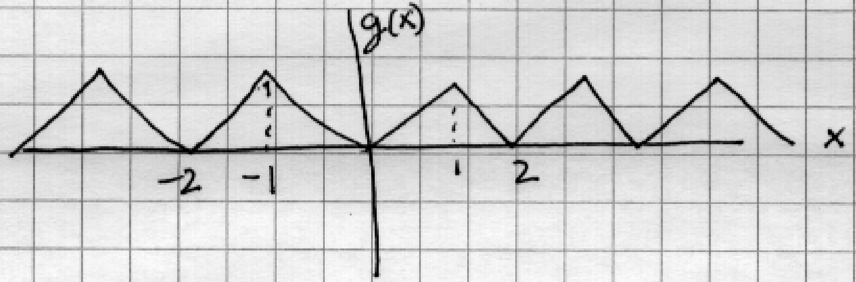
$\Rightarrow U \subset M_k$ $\because M_k$ closed

$\Rightarrow [U] \subset M_k$

(b) let $M = \bigcup_k M_k$.

Define $g: \mathbb{R} \to \mathbb{R}$ s.t. $g(x) = |x|$ for $-1 \leq x \leq 1$

and $g(x+2) = g(x)$ otherwise



By definition g is con't on $\mathbb{R}$.

$$0 \leq g(x) \leq 1 \quad \forall x$$

$$|g(x) - g(y)| \leq |x - y| \quad \forall x, y$$

Then, define

$$f_n(x) = \sum_{k=0}^{n} \left(\tfrac{3}{4}\right)^k g(4^k x)$$

Then $\underset{\text{uniformly}}{f_n \to f}$ where f is con't but not Lipschitz.

f_n is Lipschitz because

$$|f_n(x_1) - f_n(x_2)| = \left| \sum_{k=0}^{n} \left(\tfrac{3}{4}\right)^k g(4^k x_1) - g(4^k x_2) \right|$$

$$\leq \sum_{k=0}^{n} \left|\tfrac{3}{4}\right|^k |g(4^k x_1) - g(4^k x_2)|$$

$$\leq \sum_{k=0}^{n} \left|\tfrac{3}{4}\right|^k |4^k x_1 - 4^k x_2|$$

$$|f_n(x_1) - f_n(x_2)| \leq \underbrace{\left(\sum_{k=0}^{n} |3^k|\right)}_{K} |x_1 - x_2|$$

But f is not Lipschitz because.

$$\left|\frac{f(x+\delta_n) - f(x)}{\delta_n}\right| > \frac{3^n}{2}$$

$$\Leftrightarrow \quad |f(x+\delta_n) - f(x)| > \frac{3^n}{2} |\delta_n|$$

where $\delta_n = \pm \frac{4^{-n}}{2}$

refer to Gordon, R.A., "Real Analysis: a first course" p189

This shows $f_n \in M_K$ for some K.

but $f \notin M_K$ for any K

$\Rightarrow f \in M = \cup M_K$

note: since $f_n \to f$ uniformly, $f_n \to f$ in L^∞

and g is defined on $\mathbb{R}$, but given $[a,b]$

we can rescale the function to fit $(a-\delta, b+\delta)$

by a mapping $\mathbb{R} \to (a-\delta, b+\delta)$

(c) have to show $[M] = C_{[a,b]}$

($\Rightarrow$) we showed in part (a) if $f_n \in M_k$

then f is continuous on $[a,b]$

$\Rightarrow [M] \subset C_{[a,b]}$

($\Leftarrow$) if $f \in C_{[a,b]}$

$\exists\ f_n \in M_k$ s.t. $f_n \to f$.

use polynomial approximation

$\therefore\ C_{[a,b]} \subset M$.

$\square$

p. 65 #5. Suppose $A \subset R$, R a metric space.

diameter, $d(A) = \sup_{x,y \in A} \rho(x,y)$

A subset $A \subset R$ is "bounded" if $d(A)$ is finite

Have to show: union of a finite number of bounded sets is bounded.

Suppose $A_1, A_2, \ldots A_n \subset R$ s.t. $d(A_i) = m_i$

$d(A_1 \cup A_2 \ldots \cup A_n) = ? = M = \max \{m_i, \rho(A_j, A_k)\}$

where $\rho(A_j, A_k) = \sup_{\substack{x \in A_j \\ y \in A_k}} \rho(x,y)$ $\qquad 1 \le i, j, k \le n$

Let $x, y \in A_1 \cup A_2 \ldots \cup A_n$

$x \in A_j \quad y \in A_k$ for some $1 \leq j, k \leq n$

if $j = k \quad \rho(x,y) \leq m_j$

if $j \neq k \quad \rho(x,y) \leq M$.

Now $\rho(A_j, A_k)$ is finite because.

$$\rho(A_j, A_k) \leq \underbrace{\rho(A_j, 0)}_{\sup_{x \in A_j} \rho(x,0)} + \rho(0, A_k)$$

Claim: $\rho(A_j, 0)$ is finite

suppose its not.

$\exists\, x \in A_j$ s.t. $\rho(x,0) > V \quad \forall V \in \mathbb{R}$.

$\rho(x,y) \geq |\rho(x,0) - \rho(y,0)|$

$> V \quad \forall V \in \mathbb{R}$.

But $\rho(x,y) \leq m_j$

contradiction

$\Rightarrow \rho(A_j, 0)$ is finite

$\Rightarrow \rho(A_j, A_k)$ is finite

$\square$

p. 66 #8. $(\mathbb{R}, \rho)$

$$\rho(x,y) = |\arctan x - \arctan y|$$

have to show $(\mathbb{R}, \rho)$ is incomplete

To do that have to find $\{x_n\}$ that is

cauchy but does not converge to an $x \in \mathbb{R}$.

let $x_n = \tan\left(\frac{\pi}{2} - \frac{1}{n}\right)$

then $\rho(x_n, x_m) = \left| \frac{\pi}{2} - \frac{1}{n} - \left(\frac{\pi}{2} - \frac{1}{m}\right) \right|$

$$= \left| \frac{1}{m} - \frac{1}{n} \right|$$

$$\leq \left|\frac{1}{m}\right| + \left|\frac{1}{n}\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \forall\, m, n \geq N = \left\lceil \frac{2}{\varepsilon} \right\rceil$$

$$= \varepsilon$$

$\Rightarrow x_n$ cauchy.

But as $n \to \infty$ $x_n \to \infty \notin \mathbb{R}$.

p. 76. #6. conditions on p 69 (6)-(8) $\Rightarrow$

$\exists$ a fixed point

i.e. $$x_i = \sum_{j=1}^{n} a_{ij} x_j + b_i$$

$$\Leftrightarrow \quad \underline{x} = A\underline{x} + \underline{b}$$

$$A\underline{x} - \underline{x} = -\underline{b}$$

$$(A-I)\underline{x} = -\underline{b}$$

since $\underline{x}$ exist & unique $(A-I)$ is invertible.

$$\Rightarrow \quad \det(A-I) \neq 0.$$

$$\Leftrightarrow \quad \begin{vmatrix} a_{11}-1 & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-1 & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & \cdots & a_{nn}-1 \end{vmatrix} \neq 0.$$

MAT 1000 Y. Warm-up Ex. Carina Sin

\#7 (a). If A and B are disjoint closed sets,

then $A \cap B = \phi$ and $A = [A]$ and $B = [B]$

$\Rightarrow$ $[A] \cap B = \phi = A \cap [B]$

$\Rightarrow$ A and B are separated. qed

(b). If A and B are disjoint open sets,

then $A \cap B = \phi$ and $A = \operatorname{int}(A) :=$ interior of A.

Denote boundary of A by ∂A

Then $[A] = \operatorname{int}(A) \cup \partial A$.

Hence
$$\begin{aligned}
[A] \cap B &= (\operatorname{int}(A) \cup \partial A) \cap B \\
&= (A \cup \partial A) \cap B \\
&= (A \cap B) \cup (\partial A \cap B) \\
&= \phi \cup (\partial A \cap B) \\
&= \partial A \cap B
\end{aligned}$$

If $\partial A \cap B \neq \phi$, then $\partial A \cap \operatorname{int}(B) \neq \phi$

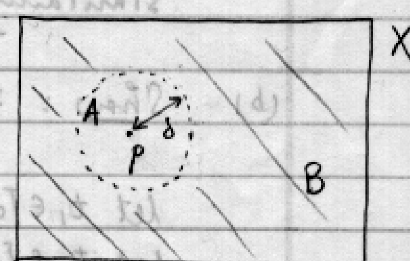
$\Rightarrow A \cap B \neq \phi$.

Therefore $[A] \cap B = \phi$.

Similarly, $A \cap [B] = \phi$. qed

(c) $A = \{q \in X \mid \rho(p,q) < \delta\}$

$B = \{q \in X \mid \rho(p,q) > \delta\}$



Then $[A] = \{q \in X \mid \rho(p,q) \leq \delta\}$.

Clearly $[A] \cap B = \phi$.

Similarly $[B] = \{q \in X \mid \rho(p,q) \geq \delta\}$

$\Rightarrow$ $A \cap [B] = \phi$.

$\therefore$ A and B are separated. qed

(d) Let X be a connected metric space with $x_1, x_2 \in X$.
Suppose $X = \{x_1, x_2, \dots, x_n, \dots\}$, $n \geq 2$
Then $X = \{x_1\} \cup \{x_2, \dots, x_n, \dots\}$
But $[\{x_1\}] \cap \{x_2, \dots, x_n, \dots\} = \{x_1\} \cap \{x_2, \dots, x_n, \dots\}$
$= \phi$.
$\Rightarrow$ $\{x_1\}$ & $\{x_2, \dots, x_n, \dots\}$ are separated.
$\Rightarrow$ $\{x_1\}$ & $X \setminus \{x_2, \dots, x_n, \dots\}$ are separated
$\Rightarrow$ X is not connected, which contradicts our assumption.
Hence X is uncountable.

qed.

#8 (a) Show: $[A_0] \cap B_0 = \phi$.

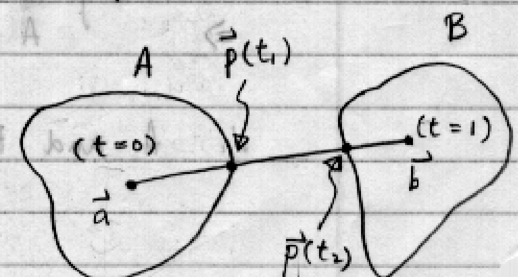
$A_0 = p^{-1}(A)$, $B_0 = p^{-1}(B)$
$[A] \cap B = \phi$
$p^{-1}([A]) \cap p^{-1}(B) = p^{-1}([A] \cap B)$
$\Rightarrow [p^{-1}(A)] \cap p^{-1}(B) = p^{-1}([A] \cap B)$
$\Rightarrow [A_0] \cap B_0 = p^{-1}(\phi) = \phi$.

Similarly $A_0 \cap [B_0] = \phi$ qed.

(b) Show: $\exists\ t_0 \in (0,1)$ s.t. $\vec{p}(t_0) \notin A \cup B$.

Let $t_1 \in [0,1]$ s.t. $\vec{p}(t_1) \in$ boundary of $A \subset [A]$
Let $t_2 \in [0,1]$ s.t. $\vec{p}(t_2) \in$ boundary of $B \subset [B]$
Since A & B are separated, $|t_1 - t_2| > 0$
Take $t_0 = \dfrac{|t_1 - t_2|}{2}$

then $t_0 \in (0,1)$
and $\vec{p}(t_0) \notin A \cup B$.



qed

#8 (c) Show: Every convex subset of $\mathbb{R}^k$ is connected.

Let S be any convex subset of $\mathbb{R}^k$.
then $\forall \vec{s}_1, \vec{s}_2 \in S, \forall t \in [0,1]$,
we have, $\vec{p}(t) = (1-t)\vec{s}_1 + t\vec{s}_2 \in S$ ———— $(*)$.
(By Definition of Convex sets)
If S is not connected,
then $\exists$ sets $A \subset S$ & $B \subset S$ s.t. $S = A \cup B$
and $[A] \cap B = A \cap [B] = \phi$.
By (b), $\exists\ t_0 \in (0,1)$ s.t. $\vec{p}(t_0) \notin A \cup B = S$
which contradicts $(*)$.
Therefore S has to be connected.

qed

#10 (a) Let X be a separable metric space
then X contains a countable dense subset (From #9).
Let $D = \{x_1, x_2, x_3, \ldots\}$ be this dense subset of X.
then $[D] = X$
Now let G be any open set of X & $x \in G \subset X = [D]$,
$\exists$ some $m \in \mathbb{N}$ s.t. $x_m \in G$.
Since G is open, $\exists$ some $n \in \mathbb{N}$ s.t. $S(x_m, \frac{1}{n}) \subset G$.
We can choose m, n s.t. $x \in S(x_m, \frac{1}{n}) \subset G$.
Then $\{S(x_m, \frac{1}{n}) \mid m, n \in \mathbb{N}\}$ is a countable base of X.

qed

(b) Let $X = \bigcup G_\alpha$ where G_α is open in X.
Since X is compact, $X = \bigcup_{i=1}^{n} G_{\alpha_i}$
Let $x \in G_{\alpha_i} \subset X$.
G_{α_i} is open $\Rightarrow G_{\alpha_i} = \bigcup S_\alpha(x_i, \varepsilon_i)$
But G_{α_i} is also compact, so $G_{\alpha_i} = \bigcup_{j=1}^{k} S_j(x_i, \varepsilon_i)$
and $x \in S_j(x_i, \varepsilon_i) \subset G_{\alpha_i}$ for some j.
Then $\{S_j(x_i, \varepsilon_i) \mid i, j \in \mathbb{N}\}$ forms a countable base

qed

#10 (c) X is a compact metric space

$\Rightarrow$ X has a countable base (By (b).)

$\Rightarrow$ X has a countable dense subset (By Thm. proved in class.)

(*) $\Rightarrow$ X is separable (By Definition in # 9)

qed

#4 (a). Show: f is continuous at all irrational points,

where $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}-\mathbb{Q} \\ \frac{1}{m} & \text{if } x \in \mathbb{Q},\ x = \frac{m}{n},\ (m,n)=1 \end{cases}$

Let $x_0 \in \mathbb{R}-\mathbb{Q}$. Need to show: $\lim_{x \to x_0} f(x) = f(x_0)$

$\forall\ \varepsilon > 0$, choose $\delta = \varepsilon$.

We can always find a sufficiently large m s.t. $x = \frac{m}{n}$ and x is close to x_0.

Then whenever $0 < |x - x_0| < \delta$,

we have
$$|f(x) - f(x_0)| = |f(x)| = \begin{cases} 0 & \text{if } x \in \mathbb{R}-\mathbb{Q} \\ \frac{1}{m} & \text{if } x = \frac{m}{n} \in \mathbb{Q}. \end{cases}$$
$$\le \frac{1}{m}$$
$$\le \delta = \varepsilon.$$

Therefore, f is continuous at all irrational points.

qed

4 (a) <u>Show</u>: f is discontinuous at all rational points.

Consider a sequence of rational numbers $\{x_n\}_{n=2}^{\infty}$:

$$\frac{100}{49}, \frac{1000}{499}, \frac{10000}{4999}, \ldots,$$

Choose $x = 2$.

Then $x_n \to x$.

Now $f(x_n) = \frac{1}{10^n}$ and $f(x) = \frac{1}{2}$.

But $\lim_{n\to\infty} f(x_n) = \lim_{x\to\infty} \frac{1}{10^n} = 0 \neq \frac{1}{2} = f(x)$.

so, $f(x_n) \not\to f(x)$ as $n \to \infty$.

Therefore, f is discontinuous at all rational points. qed.

(b) $\int_0^1 f(x)\,dx = 0$.

(c). $\forall \varepsilon > 0$, let $x_0 \in \mathbb{R}-\mathbb{Q}$.

Then choose n, m large enough so that $\frac{m}{n}$ is close to x_0.

Then $|f(x) - f(x_0)| = |f(x)|$

$$\leq \frac{1}{2^{n+m}-1} < \varepsilon.$$

$\Rightarrow f$ is continuous at all irrational numbers.

To show that f is discontinuous at rational numbers, again, consider the sequence $\{x_n\}_{n=2}^{\infty}$:

$$\frac{100}{49}, \frac{1000}{499}, \frac{10000}{4999}, \ldots, \frac{10^n}{\frac{1}{2}(10)^n-1}, \ldots$$

then $x_n \to x = 2$.

And $f(x_n) = \dfrac{1}{2^{\frac{1}{2}(10)^n-1+10^n}-1} = \dfrac{1}{2^{\frac{3}{2}(10)^n-1}-1}$,

while $f(x) = \dfrac{1}{2^{2+1}-1} = \dfrac{1}{7}$

But $\lim_{n\to\infty} f(x_n) = 0 \neq \frac{1}{7} = f(x)$ qed

Kolmogorov & Fomin's Exercise :

p66 #8

let $x_n = \tan \frac{1}{n}$ $\quad n \in \mathbb{N}$

then $\forall \varepsilon > 0$, choose $N_\varepsilon = \frac{2}{\varepsilon}$

then whenever $n, m > N_\varepsilon$,

we have $\rho(x_n, x_m) \le \rho(x_n, x_{N_\varepsilon}) + \rho(x_{N_\varepsilon}, x_m)$, By Δ Ineq.

$$= \left|\tan^{-1}(\tan \tfrac{1}{n}) - \tan^{-1}(\tan \tfrac{1}{N_\varepsilon})\right| + \left|\tan^{-1}(\tan \tfrac{1}{N_\varepsilon}) - \tan^{-1}(\tan \tfrac{1}{m})\right|$$

$$= \left|\frac{1}{N_\varepsilon} - \frac{1}{n}\right| + \left|\frac{1}{N_\varepsilon} - \frac{1}{m}\right|$$

$$= \left|\frac{\varepsilon}{2} - \frac{1}{n}\right| + \left|\frac{\varepsilon}{2} - \frac{1}{m}\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore$ $\{x_n\}$ is a Cauchy sequence.

But if we consider the distance between x_n and x, $x \in \mathbb{R}$

we have $\rho(x_n, x) = \left|\tan^{-1}(\tan \frac{1}{n}) - \tan^{-1} x\right|$

$$= \left|\frac{1}{n} - (\theta + 2n\pi)\right| \quad , \text{ where } \tan\theta = x,\ \theta \in [0, \tfrac{\pi}{2}]$$

$$\longrightarrow \infty \quad , \text{ as } n \to \infty .$$

So $x_n \not\to x$, $\forall x \in \mathbb{R}$.

Therefore this space is incomplete. qed

p 76 # 1

Take $X = (0, \frac{1}{2})$ with $\rho(x, y) = |x - y|$

Define $A : (0, \frac{1}{2}) \to (0, \frac{1}{2})$ by $Ax = x^2$

Then
$$\begin{aligned} \rho(Ax, Ay) &= |x^2 - y^2| \\ &= |x + y| \cdot |x - y| \\ &< |x - y| \quad , \because x, y \in (0, \tfrac{1}{2}) \Rightarrow |x + y| < 1 \\ &= \rho(x, y) \end{aligned}$$

But
$$\begin{aligned} Ax = x \quad &\Leftrightarrow \quad x^2 = x \\ &\Leftrightarrow \quad x^2 - x = 0 \\ &\Leftrightarrow \quad x(x - 1) = 0 \\ &\Leftrightarrow \quad x = 0 \text{ or } 1 \quad \notin X. \end{aligned}$$

So $\nexists$ a fixed point of A in space X. qed.