Warm-up Exercises.

10. Recall the p-series test from elementary calculus:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p>1 \text{ and diverges to } \infty \text{ if } p \leq 1.$$

So fix $p \in [1, \infty)$, and consider the sequence

$\left\{ \left(\frac{1}{j}\right)^{1/p} \right\}_{j=1}^{\infty}$. If $r \leq p$, then $\frac{r}{p} \leq 1$, and if $r > p$, $\frac{r}{p} > 1$.

Thus $\sum_{j=1}^{\infty} \left| \left(\frac{1}{j}\right)^{1/p} \right|^r = \sum_{j=1}^{\infty} \left(\frac{1}{j}\right)^{r/p}$ converges iff $r > p$.

Also, the sequence $\left\{ \left(\frac{1}{j}\right)^{1/p} \right\}_{j=1}^{\infty}$ is clearly bounded, so it belongs to $\ell^{\infty}(\mathbb{R}, \mathbb{N})$.

Hence the sequence belongs to $\ell^r(\mathbb{R}, \mathbb{N})$ for all $r \in (p, \infty]$, and no other r.

Next, consider the Bertrand series:

$$\sum_{n \geq 2} \frac{1}{n^{\alpha} \ln^{\beta}(n)}$$

The series converges if (i) $\alpha > 1$
or (ii) $\alpha = 1$ and $\beta > 1$

Otherwise, the series diverges.

Thus, define the sequence $\{ n^{-1/p} \ln^{-2}(n) \}_{n=2}^{\infty}$, where $p \in [1, \infty)$

Then $\sum_{n=2}^{\infty} | n^{-1/p} \ln^{-2}(n) |^r = \sum_{n=2}^{\infty} n^{-r/p} \ln^{-2r}(n)$

This Bertrand series is convergent iff $r \geq p$.
The sequence is clearly bounded, so it belongs to $\ell^{\infty}(\mathbb{R}, \mathbb{N})$.
Hence the sequence belongs to $\ell^r(\mathbb{R}, \mathbb{N})$ for

all $r \in [p, \infty]$, and no other r.

12. Suppose $\sum a_k$ converges.

Consider $2^{k-1} a_{2^k} = a_{2^k} + a_{2^k} + \cdots + a_{2^k}$

$$\leq a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k-1} + a_{2^k}$$

since $\{a_k\}$ is non-increasing.

Since $2^{k-1} a_{2^k} \leq \sum_{r=2^{k-1}+1}^{2^k} a_r$, we have

$$0 \leq \sum_{k=1}^{n} 2^{k-1} a_{2^k} \leq \sum_{k=1}^{n} \left(\sum_{r=2^{k-1}+1}^{2^k} a_r \right)$$
$$= \sum_{r=1}^{2^n} a_r$$

Thus the partial sums of the sequence $\{2^{k-1} a_{2^k}\}$ are bounded by the partial sums of $\{a_k\}$, and the partial sums are positive since every term is positive.
Since $\sum a_k$ converges, the partial sums $\sum_{k=1}^{N} a_k$ are bounded.

Hence $\lim_{n\to\infty} \sum_{k=1}^{n} 2^{k-1} a_{2^k} \leq \lim_{n\to\infty} \sum_{k=1}^{2^n} a_k < \infty$,

so $\sum 2^{k-1} a_{2^k}$ converges, since the partial sums form an increasing series that is bounded above. Of course, $2 \cdot \sum 2^{k-1} a_{2^k} = \sum 2^k a_{2^k}$ converges also.

Next, suppose $\sum 2^k a_{2^k}$ converges.

$$\sum_{r=2^{k-1}+1}^{2^k} a_r = a_{2^{k-1}+1} + \cdots + a_{2^k}$$

$$\leq a_{2^{k-1}} + a_{2^{k-1}} + \cdots + a_{2^{k-1}}$$

So $\sum_{2^{k-1}+1}^{2^k} a_r \leq 2^{k-1} a_{2^{k-1}}$,

since the a_k's are non-increasing.

The $\sum_{k=1}^{2^n} a_k = \sum_{k=1}^{n} \left(\sum_{r=2^{k-1}+1}^{2^k} a_r \right)$

$$\leq \sum_{k=1}^{n} 2^{k-1} a_{2^{k-1}}$$

Since $\sum 2^k a_{2^k}$ converges, the partial sums on the right are bounded.

Thus the partial sums $\sum_{k=1}^{2^n} a_k$ form an increasing sequence that is bounded above, and hence they converge as $n \to \infty$.

Hence $\sum a_n$ converges.

It is easy to see that the three series being considered are comprised of non-increasing sequences of positive numbers.

i) $\sum 2^k a_{2^k} = \sum 2^k 2^{-k} = \sum 1$,

which is clearly divergent. Hence $\sum \frac{1}{n}$ diverges.

ii) $\sum \frac{2^k}{2^k \ln(2^k)} = \sum \frac{1}{k \ln 2}$

$$= \frac{1}{\ln 2} \sum \frac{1}{k},$$

which is divergent, so $\sum \frac{1}{n \ln n}$ is divergent.

iii) $\sum \frac{2^k}{2^k (\ln(2^k))^2} = \sum \frac{1}{k^2 \ln^2(2)}$

$$= \frac{1}{\ln^2(2)} \sum \frac{1}{k^2}$$

This series converges, so $\sum \frac{1}{n(\ln n)^2}$ converges as well.

15. ℓ^1 metric: Suppose $x=(x_1, \ldots, x_n) \in \mathbb{R}^n$ is on the plane.

By inductive use of the triangle inequality, it is easy to see that

$$\|x-0\|_1 = |x_1| + |x_2| + \cdots + |x_n| \geq \underbrace{|x_1 + \cdots + x_n|}_{=1}$$

If $x=(1,0,0,\ldots,0)$, for example, $\|x-0\|_1 = 1$, so 1 is the shortest distance from the origin to the plane.

ℓ^2: A normal vector to the plane is $n=(1,1,\ldots,1)$. A vector from the plane to the origin is

$$v = -(x_1, x_2, \ldots, x_n), \text{ where } (x_1, \ldots, x_n) \text{ is on the plane.}$$

The distance from the plane to the origin is found by projecting n onto v:

$$\text{Distance} = \frac{|n \cdot v|}{\|n\|_2} = \frac{|x_1 + x_2 + \cdots + x_n|}{(1+1+\cdots+1)^{1/2}} = \frac{1}{\sqrt{n}}$$

ℓ^∞: $x = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ is clearly the closest point, with $\|x-0\|_\infty = 1/n$.

Otherwise, if y is on the plane, $\|y\|_\infty < \frac{1}{n}$, then $|y_1|, |y_2|, \ldots, |y_n| < 1/n$,

which implies $|y_1| + |y_2| + \cdots + |y_n| < n(\frac{1}{n}) = 1$,

contradicting $|y_1| + \cdots + |y_n| \geq \underbrace{|y_1 + \cdots + y_n|}_{=1}$ (triangle inequality)

14. By the result of question 15, the distance from the plane to the origin in $\mathbb{R}^3$ is:

ℓ^1 metric: $D = 1$

ℓ^2 metric: $D = 3^{-1/2}$

ℓ^∞ metric: $D = \frac{1}{3}$

18. Let $p \in (1, \infty)$. Let B be an ℓ^p-open ball,

$$B = B_\varepsilon^{\ell^p}(x) = \{ y \in \mathbb{R}^n \mid (\sum_1^n |x_i - y_i|^p)^{1/p} < \varepsilon \}$$

Let $\varepsilon' = \min(\varepsilon^p, 1)$.

Consider $B' = B_{\varepsilon'}^{\ell^1}(x) = \{ y \in \mathbb{R}^n \mid \sum_1^n |y_i - x_i| < \varepsilon' \}$.

Fix $y \in B'$. Note $|y_i - x_i| < 1 \ \forall\ 1 \leq i \leq n$, so

$$|y_i - x_i|^p \leq |y_i - x_i| \quad \forall\ 1 \leq i \leq n.$$

Thus $\sum_1^n |y_i - x_i|^p \leq \sum_1^n |y_i - x_i|$

$$< \varepsilon'$$
$$< \varepsilon^p,$$

hence $\|y - x\|_p < \varepsilon$, and $y \in B$.

Thus $B' \subset B$, so we can place an ℓ^1-open ball inside the ℓ^p-open balls both with centre x.

Next, let $B = B_\varepsilon^{\ell^1}(x)$, an ℓ^1-open ball.

Let $\varepsilon' = \varepsilon/n$, $B' = B_{\varepsilon'}^{\ell^p}(x)$. Let $y \in B'$.

Then $\|x - y\|_p < \varepsilon' \Rightarrow \sum_1^n |x_i - y_i|^p < (\varepsilon')^p$

Thus $|x_i - y_i|^p < (\varepsilon')^p \Rightarrow |x_i - y_i| < \varepsilon' \quad \forall\ 1 \leq i \leq n$.

Hence $\sum_1^n |x_i - y_i| < n\varepsilon'$

$$= \varepsilon,$$

so $y \in B$.

Thus we can place an ℓ^p-open ball containing x inside B.

Next, if $B = B_\varepsilon^{\ell^\infty}(x)$, let $B' = B_\varepsilon^{\ell^1}(x)$.

Then $y \in B' \Rightarrow \|x-y\|_\infty = \max_{1\le i\le n}(|x_i - y_i|)$

$$\le \sum_1^n |x_i - y_i|$$

$$= \|x-y\|_1 < \varepsilon,$$

so $y \in B$, and $B' \subset B$.

Finally, if $B = B_\varepsilon^{\ell^1}(x)$, let $\varepsilon' = \varepsilon/n$, $B' = B_{\varepsilon'}^{\ell^\infty}(x)$.

Then $y \in B' \Rightarrow \|x-y\|_1 = \sum_1^n |x_i - y_i|$

$$\le n \cdot \max_{1\le i\le n}(|x_i - y_i|)$$

$$= n\|x-y\|_\infty$$

$$< n\varepsilon' = \varepsilon,$$

so $y \in B$, and $B' \subset B$.

Thus, we have shown that if $p, q \in [1, \infty]$, and U is an open set in ℓ^p, then for any $x \in U$, we may find an ℓ^p-ball, B, $x \in B \subset U$, an ℓ^1-ball B', $x \in B' \subset B$, and an ℓ^q-ball B'', $x \in B'' \subset B' \subset B \subset U$.

Thus U is open in ℓ^q, and similarly open sets in ℓ^q are open in ℓ^p.

Thus ℓ^p is equivalent to ℓ^q $\forall\, p, q \in [1, \infty]$.

20. Suppose $x_n \to 0$ in $\ell^1(\mathbb{R}, \mathbb{N})$.

For any $\varepsilon > 0$, find $N_\varepsilon \geq 1$ so that

$$n \geq N_\varepsilon \Rightarrow \|x_n\|_1 < \varepsilon.$$

Then $n \geq N_\varepsilon \Rightarrow \|x_n\|_\infty = \max_i (|(x_n)_i|)$

$$\leq \sum_1^\infty |(x_n)_i|$$

$$= \|x_n\|_1 < \varepsilon.$$

Hence $x_n \to 0$ in $\ell^\infty(\mathbb{R}, \mathbb{N})$.

Next, let $x_1 = (1, 0, 0, \ldots)$

$x_2 = (1/2, 1/2, 0, 0, \ldots)$

$x_n = (\underbrace{1/n, 1/n, \ldots, \tfrac{1}{n}}_{n \text{ times}}, 0, 0, \ldots)$

Then $\|x_n\|_1 = 1 \quad \forall n \geq 1$,

but $\|x_n\|_\infty = \frac{1}{n} \xrightarrow[n \to \infty]{} 0$.

So $x_n \to 0$ in $\ell^\infty(\mathbb{R}, \mathbb{N})$, but not in $\ell^1(\mathbb{R}, \mathbb{N})$, so the metrics are not equivalent.

Warmup

$\boxed{6}$ Let $z = (z_1, z_2) \in \mathbb{C}^2$, with $z_j = x_j + i y_j$.

The unit ball in the ℓ^2 metric on $\mathbb{C}^2$ is those points $z \in \mathbb{C}^2$ s.t.

$$\|z\|_2^{\mathbb{C}^2} = 1, \quad \text{i.e. } \left(\sum_{j=1}^{2} |z_j|^2\right)^{1/2} = 1$$

$$= \left(\sum_{j=1}^{2} |x_j + i y_j|^2\right)^{1/2} = \left(\sum_{j=1}^{2} x_j^2 + y_j^2\right)^{1/2} = (x_1^2 + y_1^2 + x_2^2 + y_2^2)^{\frac{1}{2}}$$

$$= \|(x_1, y_1, x_2, y_2)\|_2^{\mathbb{R}^4}$$

So the ℓ^2 norm of a point in $\mathbb{C}^2$ is the same as the ℓ^2 norm of the corresponding point in $\mathbb{R}^4$. Thus, the unit balls correspond.

In ℓ^1 norm, the two unit balls do <u>not</u> correspond.

Consider $z = (0, \frac{1}{\sqrt{2}}(1+i)) \in \mathbb{C}^2$.

Considered as a point in $\mathbb{C}^2$, $\|z\|_1 = \sum_{j=1}^{2} |z_j| = |0| + |\frac{1}{\sqrt{2}}(1+i)|$

$= 1$,

So z is on the ℓ^1 unit ball in $\mathbb{C}^2$.

But the corresponding point in $\mathbb{R}^4$, $(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, is <u>not</u> on the ℓ^1 unit ball, as $\|z\|_1^{\mathbb{R}^4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} \neq 1$.

So, the ℓ^1 balls do <u>not</u> correspond.

Warmup

$\boxed{17}$ Suppose ρ, ρ' are equivalent metrics on X,
(i.e. each ε-ball in metric ρ about x_0 contains a ball in metric ρ' about x_0, and vice versa.)

Suppose $(x_n)_{n=1}^{\infty}$ converges to x_∞ in ρ metric.

Desire to show $(x_n)_{n=1}^{\infty}$ converges to x_∞ in ρ' metric.

Fix any $\varepsilon > 0$. ~~Choose N large enough so that~~

Find some $\varepsilon' > 0$ so that $B(\rho, x_\infty, \varepsilon') \subset B(\rho', x_\infty, \varepsilon)$.

Since the sequence converges in ρ, there is some N so that

$$\forall n > N \quad \rho(x_n, x_\infty) < \varepsilon', \text{ i.e. } x_n \in B(\rho, x_\infty, \varepsilon').$$

So, for $n > N$, $x_n \in B(\rho', x_\infty, \varepsilon)$. Since ε is arbitrary, the sequence converges to x_∞ in ρ' as required.

Switching the roles of ρ and ρ', we obtain "if and only if".