Topological Vectorspaces

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This little part is the first introduction to the notion of *topological vectorspace* in greatest generality. This would only be motivated after one is already acquainted with Hilbert spaces, and perhaps Banach spaces as well.

Some basic concepts are introduced which do not require the presence of a metric. Further, some concepts which would *appear* to depend upon having a metric are given sense in this more general context.

The last point is that even in this generality *finite-dimensional* topological vectorspaces have just one possible topology. This has immediate consequences for maps to and from finite-dimensional topological vectorspaces.

All this set-up works perfectly well with very mild hypotheses on the scalars involved.

After this set-up, we are ready to look at the Baire Category Theorem and pursuant results: Banach-Steinhaus (Uniform Boundedness), Open Mapping, Closed Graph, etc.

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1 First definitions: topological vectorspaces

For the moment, the 'scalars' certainly need not be the real or complex numbers, need not be locally compact, and need not even be commutative. Let k be a division ring. Any k-module V is a *free* k-module. *We will substitute 'k-vectorspace' for 'k-module' in what follows.*

Suppose that k has a **norm** $||$. That is, $||$ is a non-negative real-valued function on k so that

$$|x| = 0 \Rightarrow x = 0$$

$$|xy| = |x||y|$$

$$|x + y| \leq |x| + |y|$$

for all $x, y \in k$. Further, we suppose that with regard to the metric

$$d(x, y) = |x - y|$$

the topological space k is *complete* and *non-discrete*. The latter assertion is that, for every $\varepsilon > 0$ there is $x \in k$ so that

$$0 < |x| < \varepsilon$$

A **topological vector space** V (over k) is a k-vectorspace V with a topology on V in which *points are closed*, and so that the scalar multiplication

$$x \times v \to xv \text{ for } x \in k \text{ and } v \in V$$

and the vector addition

$$v \times w \to v + w \text{ for } v, w \in V$$

are *continuous*.

For two subsets X, Y of V, let

$$X + Y = \{x + y : x \in X, y \in Y\}$$

Also, write

$$-X = \{-x : x \in X\}$$

The following idea is completely elementary, but at the same time indispensable. Given an open neighborhood U of 0 in a topological vectorspace V, the continuity of vector addition assures that there is an open neighborhood U' of 0 so that

$$U' + U' \subset U$$

Since $0 \in U'$, necessarily $U' \subset U$ also. This can be repeated to give, for any positive integer n, an open neighborhood U_n of 0 so that

$$\underbrace{U_n + \ldots + U_n \subset U}_{n}$$

In a similar vein, for fixed $v \in V$ the map $V \to V$ by $x \to x + v$ is a *homeomorphism* as it is invertible by the obvious $x \to x - v$. Thus, *the open neighborhoods of v are of the form $v + U$ where U is an open neighborhood of* 0. In particular, giving a local basis at 0 specifies the whole topology on a topological vectorspace.

Lemma: Given a compact subset K of a topological vectorspace V, and given a closed subset C of V, if $K \cap C = \emptyset$ then there is an open neighborhood U of 0 in V so that

$$\text{closure}(K + U) \cap (C + U) = \emptyset$$

Proof: Take $x \in K$. Since C is closed, there is a neighborhood U_x of 0 so that the neighborhood $x + U_x$ of x does not meet C. By continuity of the vector addition

$$V \times V \times V \to V$$

$$(v_1, v_2, v_3) \to v_1 + v_2 + v_3$$

there is a smaller open neighborhood N_x of 0 so that

$$N_x + N_x + N_x \subset U_x$$

By replacing N_x by $N_x \cap -N_x$, which still is an open neighborhood of 0, we may also suppose that N_x is *symmetric* in the sense that $N_x = -N_x$.

Then, using this symmetry, we have

$$(x + N_x + N_x) \cap (C + N_x) = \emptyset$$

Since K is compact, there are finitely-many $x_1, \ldots, x_n$ so that

$$K \subset (x_1 + N_{x_1}) \cup \ldots \cup (x_n + N_{x_n})$$

Let

$$U = \bigcap_i N_{x_i}$$

Since the intersection is finite, this is an open set. Then

$$(K + U) \subset \bigcup_{i=1,\ldots,n} (x_i + N_{x_i} + U) \subset \bigcup_{i=1,\ldots,n} (x_i + N_{x_i} + N_{x_i})$$

All these sets are disjoint from $C + U$, by construction, since $U \subset N_{x_i}$ for all i.

Finally, since $C+U$ is a union of open sets $y+U$ for $y \in C$, it is open, so even the *closure* of $K+U$ does not meet $C+U$. ♣

Corollary: A topological vectorspace is *Hausdorff*. (Take $K=\{x\}$ and $C=\{y\}$ in the lemma). ♣

Corollary: The topological closure $\bar{E}$ of a subset E of a topological vectorspace V is obtained as

$$\bar{E} = \bigcap_U E+U$$

where U ranges over a local basis at 0.

Proof: In the lemma, take $K=\{x\}$ and $C=\bar{E}$ for a point x of V not in C. Then we obtain an open neighborhood U of 0 so that $x+U$ does not meet $\bar{E}+U$. The latter contains $E+U$, so certainly $x \notin E+U$. That is, for x not in the closure, there is an open U containing 0 so that $x \notin E+U$. This proves the assertion. ♣

It is convenient to know that Hausdorff-ness of topological vectorspaces follows from the weaker assumption that points are closed.

2 Quotients and linear maps

We continue to suppose that the 'scalars' k are a *non-discrete complete normed division ring.*

For two topological vectorspaces V, W over k, a function

$$f : V \to W$$

is **(k-)linear** if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all $\alpha, \beta \in k$ and $x, y \in V$. Almost without exception we will be interested only in **continuous linear maps**, meaning linear maps which are continuous with respect to the topologies on V, W. The **kernel** $\ker f$ of a linear map is

$$\ker f = \{v \in V : f(v) = 0\}$$

Being the inverse image of a closed set by a continuous map, it is *closed.* It is easy to check that it is a k-subspace of V.

Let H be a k-subspace of a topological vectorspace V. We wish to form the *quotient* V/H as topological vectorspace, with k-linear quotient map $q : V \to V/H$ given by

$$q : v \to v + H$$

as usual. The topology here is that a subset E of V/H is open if and only if $q^{-1}(E)$ is open. It is easy to check that this is indeed a topology: arbitrary unions and finite intersections of opens are open, as are the empty set and the whole space.

Note that the quotient map q carries open sets to open sets, so is *open*. Indeed, we have chosen the coarsest topology on the algebraic quotient V/H so that this would be so. And this topology is also the *finest* so that the quotient map q is *continuous*.

Corollary: If W is a k-subspace of a topological vectorspace V and if W is a *closed* subset of V, then the quotient V/W, with the quotient topology, is a topological vectorspace. That is, in the quotient topology points are closed.

Proof: The *algebraic* quotient exists without any topological hypotheses on W. Since W is closed, and since vector addition is a homeomorphism, $v+W$ is closed as well. Thus, its complement $V-(v+W)$ is open, so $q(V-(v+W))$ is open, by definition of the quotient topology. Thus, the complement

$$q(v) = v + W = q(v+W) = V/W - q(V-(v+W))$$

of the open set $q(V-(v+W))$ is closed. ♣

Corollary: Let $f : V \to X$ be a linear map so that $f(W) = \{0\}$. Let $\bar{f}$ be the induced map $\bar{f} : V/W \to X$ defined by $\bar{f}(v+W) = f(v)$. Then f is continuous if and only if $\bar{f}$ is continuous.

Proof: Certainly if $\bar{f}$ is continuous then $f = \bar{f} \circ q$ is continuous. The converse follows from the fact that q is *open*. ♣

That is, a continuous linear map $f : V \to X$ *factors through* any quotient V/W where W is a closed subspace contained in the kernel of f.

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3 More topological features

Now we can consider the notions of **balanced subset, absorbing subset** and also **directed set, Cauchy net**, and *completeness*. We continue to suppose that the 'scalars' k are a *non-discrete complete normed division ring.*

A subset E of V is **balanced** if for every $x \in k$ with $|x| \leq 1$ we have $xE \subset E$.

Lemma: Let U be a neighborhood of 0 in a topological vectorspace V over k. Then U contains a *balanced* neighborhood N of 0.

Proof: By continuity of scalar multiplication, there is $\varepsilon > 0$ and a neighborhood U' of $0 \in V$ so that if $|x| < \varepsilon$ and $v \in U'$ then $xv \in U$. Since k is non-discrete, there is $x_o \in k$ with $0 < |x_o| < \varepsilon$. Since scalar multiplication by a non-zero element is a homeomorphism, x_oU' is a neighborhood of 0 and $x_oU' \subset U$. Put

$$N = \bigcup_{|y|\leq 1} yx_oU'$$

Then, for $|x| \leq 1$, we have $|xx_o| \leq |x_o| < 1$, so

$$xN = \bigcup_{|y|\leq 1} x(yx_oU') = \bigcup_{|y|\leq 1} yx_oU' = N$$

This N is as desired. ♣

A subset E of a (not necessarily topological) vectorspace V over k is **absorbing** if for every $v \in V$ there is $t_o \in \mathbf{R}$ so that $v \in \alpha E$ for every $\alpha \in k$ so that $|\alpha| \geq t_o$.

Lemma: *Every neighborhood U of 0 in a topological vectorspace is absorbing.*

Proof: We may as well shrink U so as to assure that U is balanced. By continuity of the map $k \to V$ given by $\alpha \to \alpha v$, there is $\varepsilon > 0$ so that $|\alpha| < \varepsilon$ implies that $\alpha v \in U$. By the non-discreteness of k, there is non-zero $\alpha \in k$ satisfying any such inequality. Then $v \in \alpha^{-1}U$, as desired. ♣

Let S be a **poset**, that is, a set with a partial ordering $\geq$. We assume further that, given two elements $s, t \in S$, there is $z \in S$ so that $z \geq s$ and $z \geq t$. Then S is a **directed set**.

A **net** in V is a subset $\{x_s : s \in S\}$ of V indexed by a directed set S. A net $\{x_s : s \in S\}$ in a topological vectorspace V is a **Cauchy net** if, for every neighborhood U of 0 in V, there is an index s_o so that for $s, t \geq s_o$ we have $x_s - x_t \in U$. A net $\{x_s : s \in S\}$ is **convergent** if there is $x \in V$ so that, for every neighborhood U of 0 in V there is an index s_o so that for $s \geq s_o$ we have $x - x_s \in U$. Since points are closed, there can be *at most* one point to which a net converges. Thus, *a convergent net is Cauchy.* A topological vectorspace is **complete** if (also) every Cauchy net is convergent.

Lemma: Let Y be a vector subspace of a topological vector space X, and suppose that Y is *complete* when given the subspace topology from X. Then Y is a *closed* subset of X.

Proof: Let $x \in X$ be in the closure of Y. Let S be a local basis of opens at 0, where we take the partial ordering so that $U \geq U'$ if and only if $U \subset U'$. For each $U \in S$ choose

$$y_U \in (x + U) \cap Y$$

Then the net $\{y_U : U \in S\}$ converges to x, so is Cauchy. But then it must converge to a point in Y, so by uniqueness of limits of nets it must be that $x \in Y$. Thus, Y is closed. ♣

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4 Finite-dimensional spaces

Now we look at the especially simple nature of finite-dimensional topological vectorspaces, and their interactions with other topological vectorspaces as well. Still we need only suppose that the scalar field k is a complete non-discrete normed division ring. The main point of this section is that *there is only one topology on a finite-dimensional space.* This has several important consequences.

Proposition: If the topological vectorspace V is one-dimensional, i.e., is a free module on one generator e, then the map $k \to V$ given by $x \to xe$ is a *homeomorphism.*

Proof: Since scalar multiplication is continuous, we need only show that the map is *open.* Given $\varepsilon > 0$, by the non-discreteness of k there is an element x_o of k so that $0 < |x_o| < \varepsilon$. Since V is Hausdorff, there is a neighborhood U of 0 so that $x_o \notin U$. We can shrink U if necessary to be able to assume that it is *balanced.* Take $x \in k$ so that $xe \in U$. If $|x| \geq |x_o|$ then $|x_o x^{-1}| \leq 1$, so that

$$x_o e = (x_o x^{-1})(xe) \in U$$

by the balanced-ness of U, contradiction. Thus, we see that

$$xe \in U \;\Rightarrow\; |x| < |x_o| < \varepsilon$$

This proves the claim. ♣

Corollary: Fix $x_o \in k$. A not-identically-zero k-linear k-valued function f on V is *continuous* if and only if the **affine hyperplane**

$$H = \{v \in V : f(v) = x_o\}$$

is a *closed* subset of V.

Proof: Certainly if f is continuous then H is closed. For the converse, we need only consider the case $x_o = 0$, since translations (i.e., vector additions) are homeomorphisms of V to itself.

For any v_o with $f(v_o) \neq 0$ and for any other $v \in V$ we have

$$f(v - f(v)f(v_o)^{-1}v_o) = f(v) - f(v)f(v_o)^{-1}f(v_o) = 0$$

Thus, V/H is one-dimensional. Let $\bar{f} : V/H \to k$ be the induced k-linear map on V/H so that $f = \bar{f} \circ q$:

$$\bar{f}(v + H) = f(v)$$

Then $\bar{f}$ is a homeomorphism to k, by the previous result, so f is continuous. ♣

In the following theorem, the three assertions are related, and we prove them together by induction on dimension.

Theorem:

- A *finite-dimensional* vectorspace V over k has just one topology on it which makes it into a topological vectorspace over k.
- A finite-dimensional subspace V of an arbitrary topological vectorspace W over k is necessarily a *closed* subspace of W.
- A k-linear map $\phi : X \to V$ is continuous if and only if the kernel is closed.

Proof: To prove the uniqueness of the topology, it suffices to prove that for any k-basis $e_1, \ldots, e_n$ for V, the map

$$k \times \ldots \times k \to V$$

given by

$$(x_1, \ldots, x_n) \to x_1 e_1 + \ldots + x_n e_n$$

is a homeomorphism. We will prove this by induction on the dimension n, i.e., on the number of generators for V as a free k-module.

The case $n = 1$ was treated already. Granting this, we need only further note that, since k is complete, the lemma above asserting the closed-ness of complete subspaces shows that any one-dimensional subspace is necessarily closed.

Take $n > 1$. Let

$$H = ke_1 + \ldots + ke_{n-1}$$

By induction, H is closed in V, so we can form the quotient V/H. Let q be the quotient map. The space V/H is a one-dimensional topological vectorspace over k, with basis $q(e_n)$. By induction, the map

$$\phi : xq(e_n) = q(xe_n) \to x$$

is a homeomorphism to k.

Likewise, ke_n is a closed subspace and we have the quotient map

$$q' : V \to V/ke_n$$

We have a basis $q'(e_1), \ldots, q'(e_{n-1})$ for the image, and by induction the map

$$\phi' : x_1 q'(e_1) + \ldots + x_{n-1} q'(e_{n-1}) \to (x_1, \ldots, x_{n-1})$$

is a homeomorphism.

Then, invoking the induction hypothesis, the map

$$v \to (\phi \circ q)(v) \times (\phi' \circ q')(v)$$

is a continuous map to

$$k^{n-1} \times k \approx k^n$$

On the other hand, by the assumption that scalar multiplication and vector addition are continuous, the map

$$k^n \to V$$

by

$$(x_1, \ldots, x_n) \to x_1 e_1 + \ldots + x_n e_n$$

is continuous. These two maps are mutual inverses, proving that we have a homeomorphism.

Thus, a n-dimensional subspace is homeomorphic to k^n, so is complete, since (as follows readily) a finite product of complete spaces is complete. Thus, by the lemma asserting the closed-ness of complete subspaces, it is closed.

For a linear map $f : X \to k^n$ continuity certainly implies that the kernel N is closed. On the other hand, if N is closed, then X/N is a topological vectorspace of dimension at most n. Therefore, the induced map $\bar{f} : X/N \to V$ is unavoidably continuous. But then $f = \bar{f} \circ q$ is continuous, where q is the quotient map. This completes the induction step. ♣