Mat 457/1000, Term Test, March 5, 2003

There are five warm–up problems, each worth 16 points. Then, there are four harder problems, each worth 26 points. Please read all nine problems and choose five to write up.

five warm–up problems	80 points possible
four warm–problems	90 points possible
three warm–problems	100 points possible
two warm–problems	110 points possible
one warm–problem	120 points possible

I will only grade five answers! I want you to have plenty of things to choose from, but I also want you to have plenty of time to work the problems and write them up carefully. **Please do not write up more than five problems!!**
If you are worried that you are about to assume something that I want you to prove, please raise your hand and ask!

Warm-up problems

WU1: Let $C_c^\infty(\mathbb{R})$ be the test function space of infinitely differentiable compactly supported real-valued functions on $\mathbb{R}$. (Kolmogorov and Fomin would call this space "K" and refer to its members as finite functions with derivatives of all orders.) Prove that $C_c^\infty(\mathbb{R})$ has "sufficiently many" points in the sense that given two distinct regular generalized functions T_f and T_g then there is a test function $\phi \in C_c^\infty(\mathbb{R})$ so that

$$T_f(\phi) \neq T_g(\phi)$$

Recall that a generalized function is "regular" if there is a locally integrable function f so that $T(\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx$ for all $\phi \in C_c^\infty(\mathbb{R})$. NOTE: since we haven't done measure and integration yet, please assume f and g are continuous for this problem!!

WU2: Let E and F be Banach spaces. Let $\mathcal{L}(E, F)$ be the normed linear vector space of bounded linear operators from E to F.

a) Prove that $\mathcal{L}(E, F)$ is a Banach space. That is, prove it is complete.

b) Now prove that if the sequence of operators $\{A_n\} \subset \mathcal{L}(E, F)$ satisfies

$$\sum_{n=1}^{\infty} ||A_n||_{\mathcal{L}(E,F)} < \infty$$

then

$$\sum_{n=1}^{\infty} A_n := lim_{N\to\infty} \sum_{n=1}^{N} A_n \in \mathcal{L}(E, F).$$

WU3: Recall

$$l^2(\mathbb{R},\mathbb{N}) = \{\{x_k\}_1^\infty \mid \sum_{k=1}^{\infty} |x_k|^2 < \infty\}.$$

Define

$$\mathcal{D}(A) = \{\{x_k\}_1^\infty \in l^2(\mathbb{R},\mathbb{N}) \mid \sum_{k=1}^{\infty} k^2|x_k|^2 < \infty\} \subset l^2(\mathbb{R},\mathbb{N})$$

and the operator

$$A : \mathcal{D}(A) \longrightarrow l^2(\mathbb{R},\mathbb{N}) \qquad\qquad A(\{x_k\}_1^\infty) := \{kx_k\}_1^\infty.$$

a) Prove A is not a bounded operator.

b) Prove A is a closed operator. *Recall that an operator is closed if* $x_n \to x$ *and* $Ax_n \to y$ *imply* $y = Ax$.

c) Since A is closed, if $x_n \to x$ (in the l^2 metric) where $x_n \in \mathcal{D}(A)$ and $x \notin \mathcal{D}(A)$ then Ax_n *cannot* converge. Demonstrate this by presenting a sequence $\{x_n\} \subset \mathcal{D}(A)$ with limit point $x \notin \mathcal{D}(A)$ and proving that Ax_n does not converge.

d) Prove that $\mathcal{D}(A)$ is dense in $l^2(\mathbb{R},\mathbb{N})$.

e) $\mathcal{D}(A)$ is dense in $l^2(\mathbb{R},\mathbb{N})$, so why can't you just extend A to be defined on all of $l^2(\mathbb{R},\mathbb{N})$?

WU4: Let E be a Banach space and $A \in \mathcal{L}(E,E)$. Let $\{\lambda_n\}$ be a sequence in the point spectrum of A with $\lambda_n \to \lambda$. Prove that λ is in the spectrum of A. *You may not do this problem by citing the theorem that says that the spectrum of A is closed!*

WU5: Let E and F be Banach spaces and let $A \in \mathcal{L}(E,F)$.

a) Prove that if

$$||A||_{\mathcal{L}(E,F)} < 1$$

then $I - A$ is an invertible linear operator. (I.e. it is one-to-one, onto, and its inverse is in $\mathcal{L}(F,E)$.)

b) Prove that if λ is in the spectrum of A then $|\lambda| \leq ||A||_{\mathcal{L}(E,F)}$.

Harder problems

H1: Let T be a generalized function that solves the linear differential equation:

$$T' = a(x)T$$

where a is a C^∞ function. Prove that T is a classical solution of this differential equation by proving that T is a regular generalized function: $T = T_f$ where f is a C^∞ function.

H2: Let E and F be Banach spaces. Let $A \in \mathcal{L}(E, F)$ be an operator that is onto F. Prove that a small perturbation of A is still onto F. That is, prove that there exists $\epsilon > 0$ so that if $B \in \mathcal{L}(E, F)$ with $||B||_{\mathcal{L}(E,F)} < \epsilon$ then $A + B$ is onto F.

H3: Let E and F be Banach spaces and let $\{A_n\} \subset \mathcal{L}(E, F)$ be a sequence of compact operators. Assume that

$$A_n \to A \in \mathcal{L}(E, F) \qquad \text{that is, } ||A_n - A||_{\mathcal{L}(E,F)} \to 0$$

as $n \to \infty$. Prove that A is a compact operator.

H4: Let $A : l^2(\mathbb{R}, \mathbb{N}) \longrightarrow l^2(\mathbb{R}, \mathbb{N})$ be the operator defined by

$$\{Ax\}_k = \begin{cases} 0 & \text{if } k = 1 \\ \frac{x_{k-1}}{k-1} & \text{if } k > 1 \end{cases}$$

a) Prove that A is a compact operator. *You may not do this by quoting the "Hilbert-Schmidt operators are compact" theorem!*

b) Prove that the point spectrum of A is empty.

c) Why doesn't this contradict our spectral theorem for compact operators on Hilbert Spaces?