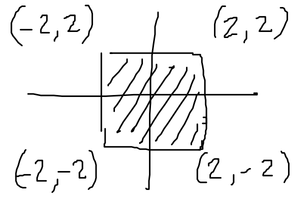
WU 1

Let L be a real vector space and $M \subset L$ be a convex body whose interior contains $\vec{0}$.

$$p_M(x) := \inf \left\{ r > 0 \;\middle|\; \frac{x}{r} \in M \right\}$$

a) $L = \mathbb{R}^2$, $M =$  find $p_M(\vec{x})$

let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ then $\frac{\vec{x}}{r} = \begin{pmatrix} x_1/r \\ x_2/r \end{pmatrix}$. We want

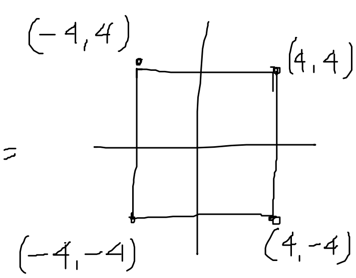
$\left| \frac{x_1}{r} \right| \leq 2$ and $\left| \frac{x_2}{r} \right| \leq 2$ to ensure $\frac{\vec{x}}{r} \in M$

$\Rightarrow \frac{|x_1|}{2} \leq r$ and $\frac{|x_2|}{2} \leq r \Rightarrow r \geq \max\left\{ \frac{|x_1|}{2}, \frac{|x_2|}{2} \right\}$

the smallest possible r that will work is

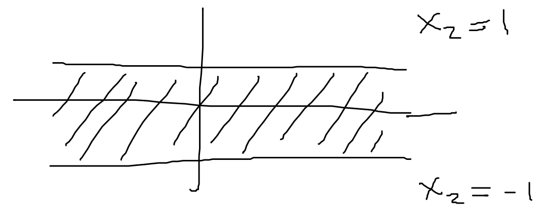
$$r_{\inf} = \frac{1}{2} \|\vec{x}\|_\infty = \frac{1}{2} \max\{|x_1|, |x_2|\}$$

$$\Rightarrow p_M(\vec{x}) = \frac{1}{2} \|\vec{x}\|_\infty$$

And $\{\vec{x} \mid p_M(\vec{x}) = 2\} = \{\vec{x} \mid \|\vec{x}\|_\infty = 4\} =$ 

WU1

b) same question, $M =$



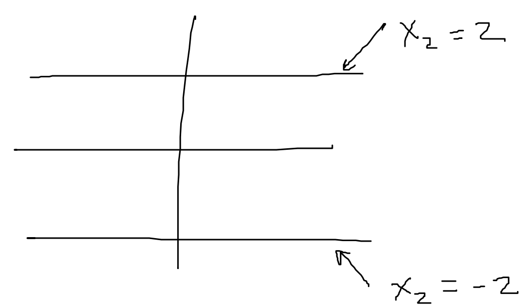
This time, we need r to satisfy

$$\left|\frac{x_2}{r}\right| \le 1 \Rightarrow |x_2| \le r \quad \text{in order for } \frac{\vec{x}}{r} \in M$$

the smallest r that will work is

$$P_M(\vec{x}) = |x_2|$$

$$\{\vec{x} \mid P_M(\vec{x}) = 2\} = \{\vec{x} \mid |x_2| = 2\}$$



c) <u>Claim</u>: $P_M(x) < \infty \quad \forall x \in L$

<u>proof</u>: fix $x \in L$. Since $\vec{0} \in I(M)$, $\exists\ \varepsilon_x > 0$ so that $|t| < \varepsilon_x \Rightarrow \vec{0} + tx \in M$. That is $tx \in M$ if $|t| < \varepsilon_x$. i.e. $\frac{x}{r} \in M$ if $r > \frac{1}{\varepsilon_x} > 0$.

This shows that $\{r > 0 \mid \frac{x}{r} \in M\}$ is nonempty.

It's bounded below by $r = 0 \Rightarrow$ the set has an infimum. This proves $P_M(x) < \infty$. //

WU1

claim: $P_M(x+y) \le P_M(x) + P_M(y)$

proof: Let $r > P_M(x)$. Then $\frac{x}{r} \in M$

Let $s > P_M(y)$. Then $\frac{y}{s} \in M$.

Since M is convex, $\alpha\left(\frac{x}{r}\right) + (1-\alpha)\left(\frac{y}{s}\right) \in M \quad \forall \alpha \in [0,1]$

Specifically, if $\alpha = \frac{r}{r+s}$. Plugging + chugging, this implies $\frac{1}{r+s}x + \frac{1}{r+s}y \in M$. i.e. $\frac{x+y}{r+s} \in M$.

this proves $r+s \ge P_M(x+y)$. This is true for all $r > P_M(x) \Rightarrow \inf\{r\} + s \ge P_M(x+y)$ Similarly, $\inf\{r\} + \inf\{s\} \ge P_M(x+y)$. But $\inf\{r\} = P_M(x)$ and $\inf\{s\} = P_M(y)$. Proving $P_M(x) + P_M(y) \ge P_M(x+y)$. //

claim: $P_M(\alpha x) = \alpha P_M(x) \quad \forall \alpha > 0$

proof: Let $r > 0$ so that $\frac{\alpha x}{r} \in M$. then $\frac{x}{r/\alpha} \in M$

$\Rightarrow \frac{r}{\alpha} \ge P_M(x) \Rightarrow r \ge \alpha P_M(x)$. Since true $\forall r > 0 \ni \frac{\alpha x}{r} \in M$, this shows $P_M(\alpha x) \ge \alpha P_M(x)$. Now, let $r > 0$ so that $\frac{x}{r} \in M$. $\Rightarrow \frac{\alpha x}{\alpha r} \in M \Rightarrow \alpha r \ge P_M(\alpha x)$

Since true $\forall r \dots$ $\alpha P_M(x) \ge P_M(\alpha x)$. Combining the two, $\alpha P_M(x) \ge P_M(\alpha x) \ge \alpha P_M(x)$ and done. //

(WU2) Let L be a real normed vector space. Fix $x_0 \in L$. Find $f \in L^*$ such that

$$f(10x_0) = 5 \quad \text{and} \quad \|f\|_{L^*} = \frac{1}{2\|x_0\|_L}$$

proof. This is a direct application of the Hahn-Banach theorem. Let $L_0 = \text{span}\{x_0\}$ be a subspace of L. We define f on the subspace L_0 and then use H-B to extend f to all of L w/o increasing its norm.

1) define f on $\text{span}\{x_0\}$. Let $y = \alpha x_0$. Define $f(y) = \frac{\alpha}{2}$. $f: L \to \mathbb{R}$ ✓ f is linear on L_0: $f(\alpha x_0 + \beta x_0) = f((\alpha+\beta)x_0) = \frac{\alpha+\beta}{2} = \frac{\alpha}{2} + \frac{\beta}{2} = f(\alpha x_0) + f(\beta x_0)$ ✓

f is a bounded functional:

$$|f(y)| = |f(\alpha x_0)| = \frac{|\alpha|}{2} = \frac{|\alpha|\|x_0\|_L}{2\|x_0\|_L} = \frac{\|y\|_L}{2\|x_0\|_L}$$

$\Rightarrow \|f\|_{L_0^*} = \frac{1}{2\|x_0\|_L}$. Now, we're done! H-B extends f to all of L and doesn't increase its norm $\Rightarrow \|f\|_{L^*} = \frac{1}{2\|x_0\|_L}$. //

WU3

Let $(L, \langle \cdot \rangle)$ be an inner-product space and let $\{\phi_k\}_1^n$ be an orthogonal family in L. Fix $f \in L$. Find $a_1 \ldots a_n$ so that $\|f - \sum_1^n a_k \phi_k\|$ is minimized.

proof: It suffices to minimize $\|f - \sum_1^n a_k \phi_k\|^2$.

$$\langle f - \sum_1^n a_k \phi_k, f - \sum_1^n a_k \phi_k \rangle = \langle f, f \rangle - \langle f, \sum_1^n a_k \phi_k \rangle - \langle \sum_1^n a_k \phi_k, f \rangle + \langle \sum_1^n a_k \phi_k, \sum_1^n a_k \phi_k \rangle$$

we'll assume L is a real space for convenience.

$$\Rightarrow \quad = \langle f, f \rangle - \sum_1^n a_k \langle f, \phi_k \rangle - \sum_1^n a_k \langle f, \phi_k \rangle + \sum_1^n \sum_1^n a_k a_l \langle \phi_k, \phi_l \rangle$$

now $\phi_k \perp \phi_L$ if $k \neq l$ so the above becomes

$$\|f - \sum_1^n a_k \phi_k\|^2 = \|f\|^2 - 2\sum_1^n a_k \langle f, \phi_k \rangle + \sum_1^n a_k^2 \|\phi_k\|^2$$

$$= \|f\|^2 + \sum_1^n a_k^2 \|\phi_k\|^2 - 2a_k \langle f, \phi_k \rangle$$

$$= \|f\|^2 + \sum_1^n \left(a_k \|\phi_k\| - \frac{\langle f, \phi_k \rangle}{\|\phi_k\|}\right)^2 - \sum_1^n \frac{\langle f, \phi_k \rangle^2}{\|\phi_k\|^2}$$

minimized if $a_k = \langle f, \phi_k \rangle / \|\phi_k\|^2$!

WU4

$\ell^1(\mathbb{R}, \mathbb{N})$ is a separable topological vector space with respect to the norm $\|\vec{x}\|_1 = \sum_1^\infty |x_i|$. Its dual is isomorphic to $\ell^\infty(\mathbb{R}, \mathbb{N})$ with the norm $\|\vec{x}\|_\infty = \sup\{|x_i|\}$

We know ℓ^∞ is not separable since it contains the following elements:

let S be a subset of $\mathbb{N}$.

define $X_S \in \ell^\infty$ by $(X_S)_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$

then there are uncountably many X_S and they're all a distance 1 from each other.

$\Rightarrow \ell^\infty$ is not separable.

WU5 Let (X, τ) be a topological vector space and X^* be the vector space of real continuous linear functionals on X.

a) define w^*. We do this by first defining a local base of open sets around $\vec{0} \in X^*$. We then use the continuity of addition to translate these open

WV5

sets to other points in X^*. This gives a base for the w* topology.

The local base at $\vec{0}$:

fix $x_1 .. x_k \in X$. Fix $\varepsilon > 0$.

$$U_{x_1..x_k;\varepsilon} = \{ \phi \in X^* \mid |\phi(x_i)| < \varepsilon \quad i = 1..k \}$$

consider all finite collections of points in X and all $\varepsilon > 0$.

b) Theorem: $f_n \xrightarrow[w*]{} f_0 \iff f_n(x) \to f_0(x) \quad \forall x \in X$

Note: it suffices to show

$$f_n - f_0 \xrightarrow[w*]{} 0 \iff f_n(x) - f_0(x) \to 0 \quad \forall x \in X$$

so if I define $\phi_n = f_n - f_0$, I just need to prove

$$\phi_n \xrightarrow[w*]{} 0 \iff \phi_n(x) \to 0 \quad \forall x \in X.$$

proof:

($\Rightarrow$) assume $\phi_n \xrightarrow[w*]{} \vec{0}$. Fix $x \in X$. Let $\varepsilon > 0$. I want to find $U \in W*$ a nbd of $\vec{0}$ so that

$\phi_n \in U \Rightarrow |\phi_n(x)| < \varepsilon$. This will prove $\phi_n(x) \to 0$. Take $U = U_{x;\varepsilon}$. U is in $w*$ and is a nbd. of $\vec{0}$. By definition of $U_{x;\varepsilon}$, $\phi_n \in U \Rightarrow |\phi_n(x)| < \varepsilon$..

($\Leftarrow$) Assume $\phi_n(x) \to 0$ $\forall x \in X$. Prove

$\phi_n \xrightarrow[w*]{} 0$. Given $U \in w^*$,

$\vec{0} \in U$, I want to find N_U so that $n \geq N_U$

implies $\phi_n \in U$. Since $\vec{0} \in U$ and $U \in w^*$,

$\exists\ x_1 .. x_k,\ \varepsilon > 0$ so that $U_{x_1 .. x_k ; \varepsilon} \subseteq U$.

Since $\phi_n(x_1) \to 0$, $\exists\ N_1$ so that $n \geq N_1 \Rightarrow |\phi_n(x_1)| < \varepsilon$

$\vdots$

Since $\phi_n(x_k) \to 0$, $\exists\ N_k$ so that $n \geq N_k \Rightarrow |\phi_n(x_k)| < \varepsilon$.

take $N_U = \max\{N_1, N_2, .. N_k\}$. By construction,

$n \geq N \Rightarrow |\phi_n(x_i)| < \varepsilon \quad i = 1..k \Rightarrow \phi_n \in U_{x_1 .. x_k ; \varepsilon}$

$\Rightarrow \phi_n \in U \Rightarrow \phi_n \xrightarrow[w*]{} \vec{0}$. //

H2 Let $L = l^2(\mathbb{C}, \mathbb{N})$ with

$$\langle x, y \rangle = \sum_1^\infty x_n \overline{y_n} \quad \text{and} \quad \|x\|_L = \sqrt{\sum_1^\infty |x_n|^2} < \infty.$$

Let L^* be the dual of L, endowed with the strong topology. Prove

$$(L^*, \|\cdot\|_{L^*}) \longleftrightarrow (L, \|\cdot\|_L)$$

proof: I do this by defining a mapping $\pi : L \to L^*$, proving it's 1:1 and onto, and proving $\|x\|_L = \|\pi(x)\|_{L^*}$.

1) define π. let $x \in L$.

define $\pi(x)$ by its action on $y \in L$.

$$\pi(x)(y) := \sum_1^\infty y_k \overline{x_k} = \langle y, x \rangle$$

$\pi(x) : L \to \mathbb{C}$?

we need to check that the series $\sum_1^\infty y_k \overline{x_k}$ converges. But this is automatic since $x \in L$ and $y \in L$. $\pi(x)$ is linear ✓ Need that $\pi(x)$ is a continuous functional. Since L has a norm, it suffices to prove $\exists\, C < \infty$ so that $|\pi(x)(y)| \leq C\|y\|$ for all $y \in L$

(#2)

$|\pi(x)(y)| = |\langle y, x\rangle| \leq \|x\| \cdot \|y\| \quad \forall y \in L$

$\Rightarrow$ $\pi(x)$ is a continuous linear functional and $\|\pi(x)\|_{L^*} \leq \|x\|_L$.

On the other hand,

$|\pi(x)(x)| = |\langle x, x\rangle| = \|x\|_L \cdot \|x\|_L$

$\Rightarrow \quad \|\pi(x)\|_{L^*} \geq \|x\|_L. \quad \Rightarrow \quad \|\pi(x)\|_{L^*} = \|x\|_L.$

Okay, we've shown that

$\pi : L \to L^* \quad$ and $\quad \|\pi(x)\|_{L^*} = \|x\|_L \quad \forall x \in L.$

π must be 1:1 since it's an isometry + linear

if $\quad \pi(x) = \pi(y) \Rightarrow \pi(x-y) = \vec{0}$

$\Rightarrow \|\pi(x-y)\|_{L^*} = \|\vec{0}\|_{L^*} = 0$

$\|$

$\|x-y\|_L \quad \Rightarrow \quad x-y = \vec{0} \Rightarrow x = y$ ✓

So all I need to do is prove that π is <u>onto</u> L^*.

#2

fix $\phi \in L^*$ I want to define $x \in L$ so that $\pi(x) = \phi$.

Let $\vec{e}_k \in L$ be $(e_k)_i = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$

define $\overline{x_k} := \phi(e_k)$. In this way, I have a sequence of complex numbers $\{x_k\}$. I want to show 1) $\{\overline{x_k}\} \in L$ and 2) $\phi(y) = \langle y, x \rangle \; \forall y \in L$

1) $\{\overline{x_n}\} \in L$.

define $y_n = \sum_1^n x_k e_k \in L$

then $|\phi(y_n)| \leq \|\phi\|_{L^*} \|y_n\|_L = \|\phi\|_{L^*} \sqrt{\sum_1^n |x_k|^2}$

On the other hand,

$|\phi(y)| = |\phi(\sum_1^n x_k e_k)| = |\sum_1^n x_k \phi(e_k)| = |\sum_1^n x_k \overline{x_k}|$

$\Rightarrow \sqrt{\sum_1^n |x_k|^2} \leq \|\phi\|_{L^*}$. this is true $\forall n$

which implies $\sum_1^\infty |x_k|^2$ converges $\Rightarrow \{\overline{x_k}\} \in L$ ✓

2) Prove that $\phi(y) = \langle y, x \rangle$ for all $y \in L$.

(this will then prove $\phi = \pi(x)$ as desired.)

fix $y \in L$. Let $y_n := \sum_1^n y_k \vec{e}_k$. We know $y_n \to y$ in L and since ϕ is continuous, $\phi(y_n) \to \phi(y)$ as $n \to \infty$.

$$\phi(y_n) = \phi\left(\sum_1^n y_k \vec{e}_k\right) = \sum_1^n y_n \overline{x_k}$$

and $\lim_{n\to\infty} \sum_1^n y_k \overline{x_k} = \sum_1^\infty y_k \overline{x_k}$ since $y \in L$, $\{\overline{x_k}\} \in L$

$$\Rightarrow \lim_{n\to\infty} \phi(y_n) = \langle y, x \rangle$$

$$\| \; \phi(y)$$

this proves $\phi(y) = \langle y, x \rangle \; \forall y \in L$, as desired.

We've now shown $\pi : L \to L^*$ is <u>onto</u> and we're done. //

(13)

<u>theorem</u>: Let (L, τ) be a locally convex topological vector space. Let $\vec{x} \in L$. if $U \in \tau$ and $x \in U$ then $\exists$ $V \in \tau$, V convex, with $\vec{x} \in V$.

<u>Note</u>: Since the translate of a convex set is convex, it suffices to prove the theorem for the special case of $\vec{x} = \vec{0}$.

<u>lemma</u>: if W is convex, then $W + W = 2W$

<u>proof</u>: let $x \in W$. then $x + x = 2x \in 2W$
$\Rightarrow 2W \subseteq W + W$.
Let $x + y \in W + W$. Then $x + y = 2\left(\frac{1}{2}x + \frac{1}{2}y\right) = 2z$
for some $z \in W$ since W convex. $\Rightarrow x + y \in 2W$. $\Rightarrow W + W \subseteq 2W$ //

<u>Proof of theorem</u>:
Let $U_1 \in \tau$ be a nbd of $\vec{0}$. Then $\exists$ $V \in \tau$, nbd of $\vec{0}$ so that $V + V \subseteq U_1$. $\exists$ $\tilde{V}$ a balanced (and absorbing) nbd of $\vec{0}$ so that $\tilde{V} \subseteq V \Rightarrow \tilde{V} + \tilde{V} \subseteq U_1$.

Since (L, τ) is locally convex, $\exists$ nonempty, convex $W \in \tau$ with $W \subset \tilde{V}$. Since $W - W$ is a convex nbd of $\vec{0}$, if we can prove $W - W \subseteq U_1$ then we're done.

W convex $\Rightarrow W = \frac{1}{2}(W + W)$ $\quad$ $W \subseteq \tilde{V}$ and $\tilde{V}$ balanced
$\Rightarrow -W = -\frac{1}{2}(W + W) \subseteq -\tilde{V} \subseteq \tilde{V}$.
$\Rightarrow \frac{1}{2}(W + W - W - W) \subseteq \tilde{V} + \tilde{V} \subseteq U_1$. $\Rightarrow \frac{1}{2}(W - W + W - W) \subseteq U_1$
$\Rightarrow W - W \subseteq U_1$ since $W - W = \frac{1}{2}(W - W + W - W)$. //

(H4) Assume (L, τ) is a real topological vector space that's 1) locally convex and 2) T_1.
Prove that given $x \neq y$ $x, y \in L$ then $\exists f \in L^*$ with $f(x) \neq f(y)$.

Note: since f is linear, it suffices to prove that given $z \in L$, $z \neq \vec{0}$, $\exists f \in L^*$ with $f(z) \neq 0$.
$(f(x-y) = f(x) - f(y) \neq 0 \Rightarrow f(x) \neq f(y).)$

Proof: fix $z \neq \vec{0}$. L is $T_1 \Rightarrow \exists U \in \tau$ with $\vec{0} \in U$, $z \notin U$. By (H3), $\exists V \subset U$, V open, V convex, $\vec{0} \in V$. I use V to define a Minkowski functional $p_V : L \to [0, \infty)$. * note: need p_V finite in order to apply H-B.

p_V is finite since V open $\Rightarrow \exists W \subset V$ open & absorbing & balanced $\Rightarrow 0 \in I(V) \neq \phi \Rightarrow V$ is a convex body.

Let $L_0 = \text{span}\{z\}$. Define f on L_0 by $f(\lambda z) = \lambda$.

f is a linear functional. Moreover, $f(y) \leq p_V(y)$ $\forall$ $y \in L_0$. Why?

Case 1: $\lambda \leq 0$ then $f(y) \leq 0$ and $p_V(y) \geq 0 \Rightarrow$ $f(y) \leq p_V(y)$

Case 2: $\lambda > 0$ then $f(y) = \lambda$ $p_V(y) = p_V(\lambda z) = \lambda p_V(z) \geq \lambda$ since $z \notin V \Rightarrow p_V(z) \geq 1$ $\Rightarrow f(y) \leq p_V(y)$ as desired.

Now extend f to all of L using Hahn Banach. Done since $f(z) = 1$. //

(F1)

<u>Claim</u>: if $(L, \|\cdot\|)$ is an infinite dimensional Banach space then $B = \{x \mid \|x\| = 1\}$ is not compact

<u>Note</u>: the proof of this is ex 3-5 from your Nov 15 homework assignment. Because I didn't give solutions to that assignment, I'll do #3-5 now.

3. Let $(L, \|\cdot\|)$ be a normed vector space and $L_0 \subsetneq L$, L_0 a closed subspace of L.

a) prove $\|x + L_0\| := \inf\{\|x+y\| \mid y \in L_0\}$ is a norm on L/L_0.

1) $$\begin{aligned} \|\alpha x + L_0\| &= \inf\{\|\alpha x + y\| \mid y \in L_0\} \\ &= \inf\{\|\alpha x + \alpha z\| \mid z \in L_0 \\ &= \inf\{|\alpha| \, \|x + z\| \mid z \in L_0\} \\ &= |\alpha| \, \|x + L\| \checkmark \end{aligned}$$

2) $$\begin{aligned} \|x+y+L\| &= \inf\{\|x+y+z\| \mid z \in L_0\} \\ &= \inf\{\|x+y+z+\tilde{z}\| \mid z, \tilde{z} \in L_0\} \\ &\le \inf\{\|x+z\| + \|y+\tilde{z}\| \mid z, \tilde{z} \in L_0\} \\ &= \inf\{\|x+z\| \mid z \in L_0\} + \inf\{\|y+\tilde{z}\| \mid \tilde{z} \in L_0\} \checkmark \end{aligned}$$

Assume $\|x + L_0\| = 0$ (show $x \in L_0$.)

$\Rightarrow \exists \{y_n\} \in L_0$ so that

$\|x + y_n\| \to 0$ as $n \to \infty$.

$\Rightarrow y_n \to x$ as $n \to \infty$

$\Rightarrow x \in [L_0]$. But L_0 is closed $\Rightarrow x \in L_0$ ✓

3b.) Prove that for any $\varepsilon > 0$ $\exists x \in L$ so that $\|x\| = 1$ and $\|x + L_0\| \geq 1 - \varepsilon$.

Proof: fix $x \notin L_0$. then $\|x + L_0\| = \inf_{y \in L_0} \|x - y\| > 0$.

Fix $\delta > 0$, Then $\exists y_\delta \in L_0$ so that

$$\|x + L_0\| \leq \|x - y_\delta\| \leq \|x + L_0\| + \delta$$

$$\Longrightarrow \frac{1}{\|x + L_0\| + \delta} \leq \frac{1}{\|x - y_\delta\|}$$

Define $\hat{x} = \frac{x - y_\delta}{\|x - y_\delta\|}$ then $\|\hat{x}\| = 1$.

Furthermore $\|\hat{x} + L_0\| = \inf_{y \in L_0} \|\hat{x} - y\| = \inf_{y \in L_0} \left\| \frac{x - y_\delta}{\|x - y_\delta\|} - y \right\|$

$$= \inf_{y \in L_0} \left\| \frac{x - y_\delta}{\|x - y_\delta\|} - \frac{y}{\|x - y_\delta\|} \right\|$$

$$= \inf_{y \in L_0} \|x - y\| \frac{1}{\|x - y_\delta\|} = \|x + L_0\| \frac{1}{\|x - y_\delta\|}$$

$$\geq \frac{\|x + L_0\|}{\|x + L_0\| + \delta} = \frac{1}{1 + \delta/\|x + L_0\|} \quad \text{as } \delta \downarrow 0 \quad \frac{1}{1 + \frac{\delta}{\|x + L_0\|}} \uparrow 1.$$

//

3c) prove that

$$\pi : L \to L/L_0$$

has norm 1

let $\mathcal{L}(L, L/L_0)$ be the space of bounded linear fxns from L to L/L_0 w/ usual norm.

<u>proof</u>.

1) $$\|\pi(x)\|_{L/L_0} = \|x + L_0\|_{L/L_0} = \inf_{y \in L_0} \|x+y\|_L$$

$$\leq \|x\|_L = 1 \cdot \|x\|_L$$

$$\Rightarrow \|\pi\|_{\mathcal{L}(L, L/L_0)} \leq 1$$

2) fix $\varepsilon > 0$. by 3b, $\exists$ x with $\|x\| = 1$ so that $\|x + L_0\|_{L/L_0} \geq 1 - \varepsilon$. $\Rightarrow$ $\frac{\|x + L_0\|}{\|x\|} \geq 1 - \varepsilon$ this is true for all $\varepsilon > 0$

$$\Rightarrow \|\pi\|_{\mathcal{L}(L, L/L_0)} \geq 1$$

combining, $\|\pi\|_{\mathcal{L}(L, L/L_0)} = 1$

3d) Prove that if L is complete then so is L/L_0.

recall, $(L, \|\cdot\|)$ is complete if and only if $\sum_1^\infty \|a_n\|$ convergent $\Rightarrow$ $\sum_1^n a_n$ convergent. (I proved this in class.)

Assume $\sum_0^\infty \|x_k + L_0\|_{L/L_0} < \infty$. I want to prove $\sum_0^N x_k + L_0 \to x_\infty + L_0$ for some $x_\infty \in L$. This will prove L/L_0 is complete by the previous comment.

Fix $\varepsilon > 0$. $\exists y_k$ so that $\|x_k + y_k\|_L \leq \|x_k + L_0\|_{L/L_0} + \frac{\varepsilon}{2^k}$

$$\Rightarrow \sum_0^\infty \|x_k + y_k\|_L \leq \sum_0^\infty \left(\|x_k + L_0\|_{L/L_0} + \varepsilon/2^k\right) = \varepsilon + \sum_0^\infty \|x_k + L_0\|_{L/L_0}$$

Then, since L is complete, we know that

$$\sum_0^\infty x_k + y_k = x_\infty \quad \text{for some } x_\infty \in L.$$

Now, I want to show $\sum_0^N x_k + L_0 \longrightarrow x_\infty + L_0$ in L/L_0.

$$\left\| \sum_0^N x_k + L_0 - (x_\infty + L_0) \right\|_{L/L_0} \leq \left\| \sum_1^N (x_k + y_k) - x_\infty \right\|_L$$

and $\left\| \sum_1^N (x_k + y_k) - x_\infty \right\|_L \to 0$ as $N \to \infty$. This proves $\sum_0^\infty x_k + L_0 = x_\infty + L_0$ as desired. //

4a) Let $(L, \|\cdot\|)$ be a normed vector space

<u>claim:</u> if L_0 is a closed subspace and $x \in L$, $x \notin L_0$ then $\{L_0 + \alpha x \mid \alpha \in \mathbb{R} \text{ (or } \mathbb{C})\}$ is a closed subspace.

<u>Proof:</u> clearly, $\{L_0 + \alpha x \mid \alpha \in \mathbb{R} \text{ or } \mathbb{C}\}$ is a subspace. We just want to show it's closed.

Assume $x_n + \alpha_n x \to z \in L$ where $x_n \in L_0$

$\Rightarrow \{x_n + \alpha_n x\}$ is Cauchy in L.

$\Rightarrow \{x_n + \alpha_n x + L_0\} = \{\alpha_n x + L_0\}$ is Cauchy in L/L_0

$\Rightarrow$ given $\varepsilon > 0$ $\exists N$ s.t. that $m, n \geq N$

$\Rightarrow \|(\alpha_m - \alpha_n)x + L_0\|_{L/L_0} < \varepsilon$.

$\Rightarrow |\alpha_m - \alpha_n| \, \|x + L_0\|_{L/L_0} < \varepsilon$ since $\|\cdot + L_0\|_{L/L_0}$ is a norm

$\Rightarrow \{\alpha_m\}$ is Cauchy in $\mathbb{R}$ $\mathbb{R}$ is complete $\Rightarrow \alpha_m \to \alpha_\infty$ as $m \to \infty$. Thus $\alpha_m x \to \alpha_\infty x$ in L.

Thus $\{x_m\} = \{\underbrace{z_m}_{\text{convergent}} - \underbrace{\alpha_m x}_{\text{convergent}}\}$ is convergent.

Since $x_n \in L_0$ and L_0 closed, $x_n \to x_\infty \in L_0$

$\Rightarrow x_n + \alpha_n x$ converges to $x_\infty + \alpha_\infty x$. $\Rightarrow \{L_0 + \alpha x\}$ closed!

//

4b

claim: every finite dimensional subspace of L is closed. (L is a $\mathbb{R}$ or $\mathbb{C}$ vector space w/ norm.)

proof: It suffices to prove that a 1-d subspace of L is closed. Then by 4a we're done since we just add dimensions one at a time.

Let $L_0 = \text{span}\{x_0\}$ $\quad x_0 \neq \vec{0} \quad x_0 \in L$

assume $\alpha_n x_0 \to x_\infty \in L$. I want to show $x_\infty \in L_0$.

$\alpha_n x$ convergent $\Rightarrow$ $\alpha_n x$ Cauchy in L $\Rightarrow$ $\{\alpha_n\}$ Cauchy in $\mathbb{R}$ (or $\mathbb{C}$) (by argument in 4a)

$\rightarrow \alpha_n \to \alpha_\infty \in \mathbb{R}$ since $\mathbb{R}$ complete

$\Rightarrow \alpha_n x \to \alpha_\infty x \Rightarrow \alpha_\infty x = x_\infty$ since limits are unique in metric spaces $\Rightarrow x_\infty \in L_0 \Rightarrow L_0$ closed. //

5 claim. Let $(L, \| \|)$ be an infinite dimensional normed vector space. Let

$$B = \{ x \in L \mid \|x\|_L \leq 1 \}$$

Then B is not compact.

Note: we don't need L to be a Banach space!

proof: fix $x_0 \in B$. By 4b, span$\{x_0\}$ is closed, by 3b $\exists\, x_1 \in B$ with $\|x_1 + L_0\| > \frac{1}{2}$. Let $L_0 = \text{span}\{x_0, x_1\}$. by 4b L_0 is closed. by 3b $\exists\, x_2 \in B$ w/ $\|x_2 + L_0\| > \frac{1}{2}$. continue. in this way, we construct $\{x_k\} \subset L$, $\|x_k\| \in B$ and $\|x_k - x_l\| > \frac{1}{2}$ if $k > l$ $\Rightarrow$ cover $\{x_k\}$ w/ open balls of radius $= \frac{1}{4} \Rightarrow \nexists$ finite subcover. done!