Solutions to first term test.

#1. $\ell^p(\mathbb{R}, \mathbb{N})$ with $p \in [1, \infty)$ is separable.

Let $A_1 = \{ x \mid x_1 \in \mathbb{Q}, x_i = 0 \ \ i \geq 2 \}$

$A_2 = \{ x \mid x_1, x_2 \in \mathbb{Q}, \ \ x_i = 0 \ \ i \geq 3 \}$

$\vdots$

$A_n = \{ x \mid x_1, x_2, \dots x_n \in \mathbb{Q}, \ \ x_i = 0 \ \ i \geq n+1 \}$

1) $A_n \subseteq \ell^p(\mathbb{R}, \mathbb{N})$ each n.

2) A_n is countable since A_n is homeomorphic to $\mathbb{Q}^n$ and $\mathbb{Q}^n$ is countable.

3) $\bigcup_1^\infty A_n$ is the countable union of countable sets $\Rightarrow \bigcup_1^\infty A_n$ is countable.

<u>claim:</u> $\bigcup_1^\infty A_n \subseteq \ell^p(\mathbb{R}, \mathbb{N})$ is dense.

proof: fix $\varepsilon > 0$, $x \in \ell^p(\mathbb{R}, \mathbb{N})$. $\exists N_\varepsilon$ so that $\sum_{N_\varepsilon}^\infty |x_i|^p < \varepsilon$.

and since $\mathbb{Q}^{N_\varepsilon - 1}$ is dense in $\mathbb{R}^{N_\varepsilon - 1}$,

$\exists \ q \in A_{N_\varepsilon - 1}$ with

$$\sum_1^\infty |q_i - x_i|^p = \sum_1^{N_\varepsilon - 1} |q_i - x_i|^p + \sum_{N_\varepsilon}^\infty |x_i|^p < \varepsilon + \varepsilon.$$

$\Rightarrow \rho_p(q, x) < \sqrt[p]{2\varepsilon}$ and $\bigcup_1^\infty A_n$ is dense in $\ell^p(\mathbb{R}, \mathbb{N})$.

2.

proof 1:

$$\rho_2(\phi_{nx}, 0)^2 = \int_0^{2\pi} \phi_{nx}(x)\,\phi_{nx}(x)\,dx$$

$$= -\int_0^{2\pi} \phi_n(x)\,\phi_{nxx}(x)\,dx + \phi_n \phi_{nx}\Big|_0^{2\pi}$$

$$= -\int_0^{2\pi} \phi_n(x)\,\phi_{nxx}(x)\,dx \quad \text{since}$$

$$\phi_n(2\pi) = \phi_n(0)$$

$$\phi_{nxx}(2\pi) = \phi_{nxx}(0)$$

$$\Rightarrow \rho_2(\phi_{nx}, 0)^2 \le \int_0^{2\pi} |\phi_n(x)|\,|\phi_{nxx}(x)|\,dx$$

$$\le \sqrt{\int_0^{2\pi} \phi_n(x)^2\,dx}\,\sqrt{\int_0^{2\pi} \phi_{nxx}(x)^2\,dx}$$

$$\le 8\sqrt{\int_0^{2\pi} \phi_n(x)^2\,dx}$$

$$< 8\varepsilon \quad \text{if } n \ge N_\varepsilon$$

$$\text{since } \rho_2(\phi_n, 0) \to 0$$

$$\Rightarrow \rho_2(\phi_{nx}, 0) \to 0 \quad \text{as desired}$$

Proof 2

since ϕ_n is continuous and periodic,

$$\phi_n(x) = \sum_{-\infty}^{\infty} \hat{\phi}_n(k) e^{ikx}$$

for $\hat{\phi}_n(k) \in \mathbb{C}$.

and
$$\phi_{nx}(x) = \sum_{-\infty}^{\infty} ik\hat{\phi}_n(k) e^{ikx}$$

$$\phi_{nxx}(x) = \sum_{-\infty}^{\infty} -k^2\hat{\phi}_n(k) e^{ikx}$$

$$\Rightarrow \rho(\phi_n, 0) = \sqrt{\sum_{-\infty}^{\infty} |\hat{\phi}_n(k)|^2}$$

$$\rho(\phi_{nx}, 0) = \sqrt{\sum_{-\infty}^{\infty} k^2|\hat{\phi}_n(k)|^2}$$

$$\rho(\phi_{nxx}, 0) = \sqrt{\sum_{-\infty}^{\infty} k^4|\hat{\phi}_n(k)|^2}$$

now we're done by Cauchy-Schwartz on $\ell^2(\mathbb{C}, \mathbb{Z})$.

$$\rho(\phi_{nx}, 0)^2 = \sum_{-\infty}^{\infty} k^2|\hat{\phi}_n(k)|^2 = \sum_{-\infty}^{\infty} k^2|\hat{\phi}_n(k)|\,|\hat{\phi}_n(k)|$$

$$\leq \sqrt{\sum k^4|\hat{\phi}_n(k)|^2}\sqrt{\sum |\hat{\phi}_n(k)|^2}$$

$$\leq 8 \cdot \varepsilon \quad \text{if } n \geq N_\varepsilon \text{ since } \rho_2(\phi_n, 0) \to 0. \quad //$$

problem # 3

Fix $\varepsilon > 0$. Since K is continuous on $[0,1] \times [0,1]$ which is a compact set, K is uniformly cont.

$\Rightarrow \exists\, \delta > 0$ s.t. that

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} < \delta \Rightarrow |k(x_1, y_1) - k(x_2, y_2)| < \varepsilon.$$

Specifically,

$$|x_1 - x_2| < \delta \Rightarrow |K(x_1, y) - K(x_2, y)| < \varepsilon.$$

1) Tf is continuous

$$|Tf(x_1) - Tf(x_2)| = \left| \int K(x_1, y) f(y)\,dy - \int K(x_2, y) f(y)\,dy \right|$$

$$\leq \int_0^1 |K(x_1, y) - K(x_2, y)|\,|f(y)|\,dy$$

$$\leq \rho_\infty(f, 0) \int_0^1 |K(x_1, y) - K(x_2, y)|\,dy$$

$$< \rho_\infty(f, 0) \int_0^1 \varepsilon\,dy \quad \text{if } |x_1 - x_2| < \delta$$

$$= \rho_\infty(f, 0)\,\varepsilon.$$

$$\Rightarrow \quad |x_1 - x_2| < \delta \Rightarrow |Tf(x_1) - Tf(x_2)| < \rho_\infty(f, 0)\,\varepsilon.$$

Since f is fixed, $\rho_\infty(f, 0)$ is some fixed number $\Rightarrow$ we've proven Tf is continuous on $[0,1]$.

Now consider the set

$A = \{ Tf \mid \rho_\infty(f, 0) \leq 1 \}$

<u>claim</u>: A is relatively compact in $C([0,1])$

<u>proof</u>: it suffices to show that A is a uniformly bounded, equicontinuous family of functions. (Arzela-Ascoli.)

1) equicontinuity.

Let f be such that $\rho_\infty(f, 0) \leq 1$

then $|x_1 - x_2| < \delta$

$$\Rightarrow |Tf(x_1) - Tf(x_2)| < 1 \cdot \varepsilon = \varepsilon$$

by calculation before.

$\Rightarrow$ we've found a δ that works for all $Tf \in A$ and all $x_1 \in [0,1]$.

$\Rightarrow$ A is equicont

2) uniform bound

$$|Tf(x)| \leq \int |K(x,y)| |f(y)| \, dy$$

$$\leq \rho_\infty(f, 0) \int |K(x,y)| \, dy$$

$$\leq \int |K(x,y)| \, dy$$

$\Rightarrow$ $|Tf(x)| \leq$ some fixed # $\Rightarrow$ $\rho_\infty(Tf, 0) \leq C < \infty$. //

Problem 4:

take $X = [1, \infty)$ with the usual metric.

Let $Ax = x + 1/x$

then Ax is a contraction since its derivative is $1 - 1/x^2 < 1$ if $x \geq 1$

and $|Ax - Ay| = |1 - 1/\xi^2|\,|x-y| < |x-y|$

where $\xi \in (x, y)$ (mean value theorem)

But $\nexists$ $x_\infty \in [2, \infty)$ so that

$x_\infty = Ax_\infty$ because this would imply $1/x_\infty = 0$.

This solution is attributed to a pair of 3^{rd} year math students: Stephen Green and Geordie Richards.

Theorem: Let (X, d) be a compact metric space and $F: X \to X$ satisfy
$$d(F(x), F(y)) < d(x, y) \quad \text{if } x \neq y$$
Then F has a fixed point (which is unique)

proof:
uniqueness. Assume $F(x) = x$ and $F(y) = y$ but $x \neq y$. Then $d(F(x), F(y)) < d(x, y)$ (since $x \neq y$). $\Rightarrow d(x, y) < d(x, y)$. Which is impossible. So $\nexists$ two fixed points.

existence: F is a continuous function (just take $\varepsilon = \delta$). $\Rightarrow G(x) := d(x, F(x))$ is a continuous function. Since X is a compact set, G achieves its minimum on X. i.e.
$\exists\, x_0 \in X$ such that $\quad d(x_0, F(x_0)) \leq d(y, F(y))$
for all $y \in X$.

Suppose $G(x_0) > 0$. then $F(x_0) \neq x_0$

Then
$$\begin{aligned} G(F(x_0)) &= d(F(x_0), F(F(x_0))) \\ &< d(x_0, F(x_0)) \quad \text{(by contraction)} \\ &= G(x_0). \end{aligned}$$

thus $G(F(x_0)) < G(x_0)$ contradicting that G achieves its minimum at x_0. Therefore $G(x_0) = 0$, and hence $F(x_0) = x_0$.

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Note: You cannot conclude from "X compact" and "$d(F(x),F(y)) < d(x,y)$ for $x \neq y$" that $\exists\, \alpha < 1$ such that $d(F(x),F(y)) \leq \alpha\, d(x,y)$.

Here's why. Consider $F(x) = \sin(x)$ on $X = [-\pi/2, \pi/2]$.

First of all, we know

$|\sin(x) - \sin(y)| < |x-y|$ because

$$\sin(x) - \sin(y) = \int_x^y \cos(z)\, dz$$

$$\rightarrow |\sin(x) - \sin(y)| \leq \int_x^y |\cos(z)|\, dz < \int_x^y 1\, dz = |x-y|$$

(argument due to Alex Bloemendal).

Also, we know $\nexists\, \alpha < 1$ such that

$|\sin(x) - \sin(y)| \leq \alpha |x-y|$ since

$$\alpha = \sup_x \sup_{y \neq x} \frac{|\sin(x) - \sin(y)|}{|x-y|}$$

$$\geq \sup_x \frac{|\sin(x) - \sin(-x)|}{|x-(-x)|} = \sup_x \frac{|\sin(x)|}{|x|} = 1.$$