Mat 457/1000, Term Test, October 28, 2002

Each problem is worth 25 points. You may assume basic facts from integration — you do not have to work with Reimann sums below. Write out your proofs/counterexamples fully and clearly. If you’re worried that you’re about to assume something I want you to prove, raise your hand and ask!!

1. Consider \( L^p(\mathbb{R}, \mathbb{N}) \), the set of bounded sequences of real numbers that satisfy

\[
\sum_{n=1}^{\infty} |x_n|^p < \infty.
\]

Prove or disprove: if \( p \in [1, \infty) \) then \( L^p(\mathbb{R}, \mathbb{N}) \) is separable.

2. Let \( (X, \rho) \) be the space of continuous real-valued functions on \([0, 2\pi]\) with the metric

\[
\rho_2(f, g) = \sqrt{\int_0^{2\pi} |f(x) - g(x)|^2 \, dx}.
\]

Let \( X_0 \subset X \) be those functions that satisfy the additional conditions: \( \phi \in X_0 \) if \( \phi \) is periodic, \( \phi_x \in X \), and \( \phi_{xx} \in X \). That is, \( \phi \) is periodic, has two derivatives (each periodic), and each derivative is continuous on \([0, 2\pi]\).

Let \( \{\phi_n\} \) be a sequence in \( X_0 \) that satisfies: \( \lim_{n \to \infty} \rho_2(\phi_n, 0) = 0 \) and \( \rho_2(\phi_{nx}, 0) \leq 8 \).

Prove \( \lim_{n \to \infty} \rho_2(\phi_{nx}, 0) = 0 \)

3. Let \( K \in C([0, 1] \times [0, 1]) \). That is, the space of continuous real-valued functions with the metric

\[
\rho_\infty(K, G) = \sup_{(x, y) \in [0, 1] \times [0, 1]} |K(x, y) - G(x, y)|
\]

Given \( f \in C([0, 1]) \), define the real-valued function \( T f \) as follows:

\[
T f(x) = \int_0^1 K(x, y) f(y) \, dy.
\]

Prove that \( T f \in C([0, 1]) \) and that the set \( \{T f \mid \rho_\infty(f, 0) \leq 1\} \) is relatively compact in \( C([0, 1]) \).

4. Let \( A \) be a mapping from a complete metric space \( (X, \rho) \) into itself. Prove that the condition

\[
\rho(Ax, Ay) < \rho(x, y) \quad (x \neq y)
\]

does not imply that \( A \) has a fixed point, but if \( X \) is compact in addition to being complete then it does imply that \( A \) has a fixed point. (For full credit, you’ll have to re-prove the fixed point theorem from Kolmogorov and Fomin as part of your answer.)