Mat 457/1000, Term Test, October 28, 2002

Each problem is worth 25 points. You may assume basic facts from integration — you do not have to work with Reimann sums below. Write out your proofs/counterexamples fully and clearly. If you're worried that you're about to assume something I want you to prove, raise your hand and ask!!

1. Consider $l^p(\mathbb{R}, \mathbb{N})$, the set of bounded sequences of real numbers that satisfy

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

   Prove or disprove: if $p \in [1, \infty)$ then $l^p(\mathbb{R}, \mathbb{N})$ is separable.

2. Let (X, ρ) be the space of continuous real-valued functions on $[0, 2\pi]$ with the metric

$$\rho_2(f, g) = \sqrt{\int_0^{2\pi} |f(x) - g(x)|^2 \, dx}.$$

   Let $X_0 \subset X$ be those functions that satisfy the additional conditions: $\phi \in X_0$ if ϕ is periodic, $\phi_x \in X$, and $\phi_{xx} \in X$. That is, ϕ is periodic, has two derivatives (each periodic), and each derivative is continuous on $[0, 2\pi]$.

   Let $\{\phi_n\}$ be a sequence in X_0 that satisfies: $\lim_{n\to\infty} \rho_2(\phi_n, 0) = 0$ and $\rho_2(\phi_{nxx}, 0) \leq 8$. Prove $\lim_{n\to\infty} \rho_2(\phi_{nx}, 0) = 0$

3. Let $K \in C([0, 1] \times [0, 1])$. That is, the space of continuous real-valued functions with the metric

$$\rho_\infty(K, G) = \sup_{(x,y)\in[0,1]\times[0,1]} |K(x, y) - G(x, y)|$$

   Given $f \in C([0, 1])$, define the real-valued function Tf as follows:

$$Tf(x) = \int_0^1 K(x, y) f(y) \, dy.$$

   Prove that $Tf \in C([0, 1])$ and that the set $\{Tf \mid \rho_\infty(f, 0) \leq 1\}$ is relatively compact in $C([0, 1])$.

4. Let A be a mapping from a complete metric space (X, ρ) into itself. Prove that the condition

$$\rho(Ax, Ay) < \rho(x, y) \qquad (x \neq y)$$

   does not imply that A has a fixed point, but if X is *compact* in addition to being complete then it does imply that A has a fixed point. (For full credit, you'll have to re-prove the fixed point theorem from Kolmogorov and Fomin as part of your answer.)