Mat457Y/Mat1000Y Homework, due Friday November 15, 2002

Homework Problems

1. a) Consider X_∞, the vector space of continuous real-valued periodic functions on $[a,b]$ with the norm $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$. Prove that the set of trigonometric polynomials

$$p(x) = \sum_{k=0}^{N} a_k \cos\left(k\frac{2\pi x}{b-a}\right) + b_k \sin\left(k\frac{2\pi x}{b-a}\right) \qquad \text{with } a_k, b_k \in \mathbb{R}$$

is dense in X_∞.

b) Consider X_2, the vector space of continuous real-valued functions on $[a,b]$ with the inner product

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx.$$

Prove that the above trigonometric polynomials are also dense in X_2. (Hint: use 1a.)

c) Consider X_p, the vector space of continuous real-valued functions on $[a,b]$ with the norm

$$||f-g|| = \sqrt[p]{\int_a^b |f(x)-g(x)|^p\,dx} \qquad p \in [1,\infty).$$

Prove that the above trigonometric polynomials are also dense in X_p. (Hint: use 1a.)

d) Now, consider Y_∞, the vector space of continuous complex-valued periodic functions on $[a,b]$ with the norm $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$. Prove that the set of trigonometric polynomials

$$q(x) = \sum_{k=0}^{N} c_k e^{ik\frac{2\pi x}{b-a}} \qquad \text{with } c_k \in \mathbb{C}$$

is dense in Y_∞. (Hint: use 1a)

e) Consider Y_2, the vector space of continuous complex-valued functions on $[a,b]$ with the inner product

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx.$$

Prove that the above trigonometric polynomials are also dense in Y_2. (Hint: use 1a)

f) Consider Y_p, the vector space of continuous complex-valued functions on $[a,b]$ with the norm

$$||f-g|| = \sqrt[p]{\int_a^b |f(x)-g(x)|^p\,dx} \qquad p \in [1,\infty).$$

Prove that the above trigonometric polynomials are also dense in Y_p. (Hint: use 1a.)

g) Now think about the space of bounded continuous real-valued functions on $(-\infty,\infty)$ with the L^∞ norm. Given a function f, how might you try to approximate f with known functions? What concerns would you have? What are some constraint(s) that you might put on f to assuage some of your concerns? *Don't prove anything, just write a paragraph or two that demonstrate some fundamental differences between the case of $C((-\infty,\infty))$ and $C([a,b])$.*

2. Let X be the space of continuous functions on the half-open interval $[a,b)$ with the sup-norm

$$||f-g||_\infty = \sup_{x\in[a,b)} |f(x)-g(x)|.$$

a) Prove that $(X, ||\cdot||)$ is not separable.
b) Reconcile this with the Stone Weierstrauss theorem, which states that $C([a,b])$ is separable.
c) Prove that the unit ball in X is not compact by constructing an infinite set that has no limit point.

3. Let $(L, ||\cdot||)$ be a normed vector space and L_0 a proper closed subspace of L.
a) prove that

$$||x+L_0|| := \inf\{||x+y|| \text{ where } y \in L_0\}$$

is a norm on L/L_0.
b) Prove that for any $\epsilon > 0$, there exists $x \in L$ such that $||x|| = 1$ and $||x+L_0|| \geq 1-\epsilon$.
c) Prove that the projection map $\pi(x) = x + L_0$ from L to L/L_0 has norm 1.
d) Prove that if L is complete then so is L/L_0.

4. Let $(L, ||\cdot||)$ be a normed vector space.
a) Prove that if L_0 is a closed subspace and $x \in L, x \notin L_0$ then $\{L_0 + \alpha x | \alpha \in \mathbb{R} \text{ or } \mathbb{C}\}$ is closed.
b) Prove that every finite-dimensional subspace of L is closed.

5. Let $(L, ||\cdot||)$ be an infinite dimensional normed vector space. Consider the closed unit ball

$$B = \{x \in L | ||x|| \leq 1\}$$

Prove that B is not compact. (Hint: construct a sequence of vectors such that $||x_j|| = 1$ for all j and such that $||x_j - x_k|| \geq 1/2$ for all $j \neq k$.)

Note: this is the reason theorems like Arzela Ascoli are so valuable! There, you have a family of uniformly bounded functions (i.e. they live in a closed ball in $C([a,b])$) and so there's no reason for them to have a convergent subsequence, since the closed ball isn't compact. Arezela Ascoli tells you that if you have additional information, like the family is equicontinuous, then there will be a convergent subsequence after all. Similarly, if you look at a sequence in $l^2(\mathbb{R},\mathbb{N})$ that satisfies $||x_n|| \leq 1$, there's no reason that you'd have a convergent subsequence, since the unit ball isn't compact. But if you have additional information, like each member of the sequence satisfies $\sum x_{n\,k}\, k^{1/10} < \infty$, then there will be a convergent subsequence after all. Of course, such sequences are nicer than random members of the unit ball in l^2 and that's why we get that compactness. Or if you look at the functions in $C([0,1])$ with $||f||_\infty \leq 1$, there's no reason that you'd have a convergent subsequence, since the unit ball isn't compact. But if you look at their image under the mapping T where $Tf(x) = \int K(x,y)f(y)\,dy$ then the image of the unit ball will be compact if K is a nice enough kernel. (Just think about it — if K has one derivative in x then Tf will have a derivative, for example. Clearly Tf is nicer than f.)