Mat457Y/Mat1000Y Homework Problems, due Wednesday October 16, 2002

1. a) Prove $f : (X, \mathcal{T}) \longrightarrow \mathbb{R}$ is continuous at y if and only if it is both upper and lower semi-continuous at y.
b) Show that if f and g are lower semincontinuous functions then so are $f + g$ and $f \vee g$ (where $f \vee g(x) = \max(f(x), g(x))$).
c) Let $\{f_n\}$ be a sequence of lower semicontinuous functions. Show that f defined by $f(x) = \sup_n f_n(x)$ is also lower semicontinuous.
d) A real-valued function ϕ defined on $[a, b]$ is called a step function if there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that for each i the function assumes only one value in the interval (x_i, x_{i+1}). Show that a step function ϕ is lower semicontinuous if and only if $\phi(x_i)$ is less than or equal to the smaller of the two values assumed in (x_{i-1}, x_i) and (x_i, x_{i+1}).
e) Prove that a function f defined on $[a, b]$ is lower semicontinuous if and only if there is a monotone increasing sequence $\{\phi_n\}$ of lower semicontinuous step functions on $[a, b]$ such that for each x we have $\lim_{n\to\infty} \phi_n(x) = f(x)$.
f) Prove that a function f defined on $[a, b]$ is lower semicontinuous if and only if there is a monotone increasing sequence $\{\psi_n\}$ of continuous functions such that for each x $\lim_{n\to\infty} \psi_n(x) = f(x)$.

2. Consider the following definition: *Let $\mathcal{G}^0$ be a collection of subsets of X with the following property. The collection $\mathcal{G}$ of all finite intersections of members of $\mathcal{G}^0$ satisfies: 1) for each $x \in X$, there is some $G \in \mathcal{G}$ with $x \in G$, and 2) if G_α and G_β are in $\mathcal{G}$ and $x \in G_\alpha \cap G_\beta$ then there is a $G_\gamma \in \mathcal{G}$ with $x \in G_\gamma \subset G_\alpha \cap G_\beta$. Then we call $\mathcal{G}^0$ a subbase for the topology generated by $\mathcal{G}$.*

   Proposition: 1) Let $(X, \mathcal{T})$ be a topological space and let p be a function from X to a set Y. The set $\mathcal{T}'$ of subsets of Y defined by

   $$U \in \mathcal{T}' \leftrightarrow p^{-1}(U) \in \mathcal{T}$$

   is a topology on Y, and p is a continuous function from $(X, \mathcal{T})$ to $(Y, \mathcal{T}')$.
   2) Let $(Y, \mathcal{T}')$ be a topological space and let $\mathcal{F}$ be a a set of functions from a set X to Y. The set $\mathcal{G}^0$ of subsets defined by

   $$U \in \mathcal{G}^0 \leftrightarrow U = f^{-1}(V) \qquad \text{for some } f \in \mathcal{F} \text{ and some } V \in \mathcal{T}'$$

   is a subbase for a topology $\mathcal{T}$ on X, and all functions $f \in \mathcal{F}$ are continuous from $(X, \mathcal{T})$ to $(Y, \mathcal{T}')$.

   a) Prove the proposition.
   b) Let $X = \mathbb{R}^2$ and let $(Y, \mathcal{T}')$ be the real line with its usual topology. For each $\vec{a} \in X$, define the function

   $$f_{\vec{a}} : X \longrightarrow Y \qquad \text{by} \qquad f_{\vec{a}}(\vec{x}) = \langle \vec{a}, \vec{x} \rangle.$$

   Let $\mathcal{F} = \{F_{\vec{a}} | \vec{a} \in \mathbb{R}^2\}$ and define a topology $\mathcal{T}_w$ on $\mathbb{R}^2$ as in part b of the proposition. This is called the "weak topology" on $\mathbb{R}^2$. Prove that the weak topology equals the usual metric topology on $\mathbb{R}^2$.
   c) Could you have generated the same weak topology using a smaller family of functions? If so, give me the family and prove it works.
   d) Generalize your results to $\mathbb{R}^n$.

3. Let X be the set of continuous functions on $[0,1]$ and let $\mathcal{T}$ be the topology generated by the L^2 metric

$$\rho(f,g) = \sqrt{\int_0^1 (f(x)-g(x))^2\,dx}.$$

For $a \in X$ we define a function $F_a : X \longrightarrow \mathbb{R}$ by

$$F_a(f) = \int_0^1 f(x)a(x)\,dx.$$

Assume the usual metric topology on $\mathbb{R}$ and let $\mathcal{T}_w$ be the weak topology induced on X by the family of functions $\mathcal{F} = \{F_a | a \in X\}$ as in part b of the proposition.

a) Prove that a sequence $\{f_n\} \subset X$ converges weakly to f if and only if

$$\lim_{n\to\infty} \int_0^1 (f_n(x) - f(x))a(x)\,dx = 0$$

for all $a \in X$.
b) Prove that if f_n converges to f in the metric topology $\mathcal{T}$ then it converges to f in the weak topology $\mathcal{T}_w$.
c) Find a sequence $\{f_n\}$ that converges to f in the weak topology but does not converge in the metric toplogy $\mathcal{T}$. Does this imply $\mathcal{T}$ has more open sets than $\mathcal{T}_w$ or fewer?

4. Let Ω be the set of continuous real-valued functions ω on $[0,\infty)$ satisfying $w(0) = 0$. We introduce a topology on Ω via a subbase $\mathcal{G}^0$. The basic open sets are:

$$U(t_0, a, b) = \{\omega | \omega(t_0) \in (a,b)\}$$

where $t_0 > 0$ and $-\infty \leq a < b \leq \infty$.

a) Verify that the collection of all sets $U(t_0, a, b)$ is a subbase.
b) Is $A_{t_0} = \{\omega | \omega(t_0) \neq 0\}$ an open set? A closed set?
c) Is $B_T = \{\omega | \omega(t) \neq 0 \forall 0 < t < T\}$ an open set? A closed set? Does it have any interior points?
d) Is $C_T = \{\omega | \omega(t) = 0 \quad \text{for some} \quad 0 < t < T\}$ an open set? A closed set?

5. Consider the following definition: *Let $(X, \mathcal{T})$ be a topological space. A G_δ set is a countable intersection of open sets, and an F_σ set is a countable union of closed sets. A countable intersection of F_σ sets is denoted by $F_{\sigma\delta}$, and a countable union of G_δ sets is denoted by $G_{\delta\sigma}$. Likewise, $G_{\delta\sigma\delta}$ is a countable intersection of a countable union of G_δ sets and so on.*

a) Prove that the complement of a G_δ set is a F_σ set and vice versa.
b) Prove that B_T is a $G_{\delta\sigma\delta}$ set.