Mat457Y/Mat1000Y Warm-up exercises and Homework Problems, due Friday September 27, 2002

Warm-up exercises

The two of you who're writing up warm-up exercises should write up the following eleven problems: my 4, 7, 8, and 10. Kolmogorov and Fomin's p. 54, #5; p. 55, #11; p. 65, #5; p. 66, #8; p. 76, #1; p. 76, #3; p. 76, #6.

1. For each of the following functions, find the set of $p \in [1, \infty]$ for which $f \in L^p$ on the given interval.
   a) $f(x) = \exp(-x)$ on $[0, \infty)$
   b) $f(x) = 1/x$ on $[1, \infty)$
   c) $f(x) = 1/\sqrt{x}$ on $(0, 1]$
   d) $f(x) = \ln(x)$ on $(0, 3]$
   e) $f(x) = 1/sqrtx$ on $(0, \infty)$
   f) $f(x) = 1/sqrt|x|$ on $0 < |x| \leq 1$, $= 0$ at $x = 0$, and $= 2/x^2$ on $|x| > 1$.

2. In $L^2([0, 2\pi])$, find parameters α, β, and γ that minimize the distance between $f(x) = \sin(x)$ and
   a) the constant function $g_0(x) = \alpha$.
   b) the linear function $g_1(x) = \alpha + \beta x$.
   c) the quadratic function $g_2(x) = \alpha + \beta x + \gamma x^2$.
   (You can choose different parameters for each question.)

3. Suppose that
$$\int_0^\infty |f(x)|\, dx < \infty.$$
   Prove, or give a counter-example: $\lim_{x \to \infty} |f(x)| = 0$.

4. Let $f(x) = 0$ if x is irrational and $= 1/m$ if x is rational and $x = m/n$ where m and n have no common factors.
   a) Show that f is continuous at all irrational points and discontinuous at all rational points.
   b) Find the Riemann integral of f over $[0, 1]$. (Yes, I want you to go in and get your hands dirty with the Riemann sums!)
   c) If, instead of $f(x) = 1/m$, we have $f(x) = 1/(2^{n+m} - 1)$ at the rationals, show that f is still continuous at the irrational points and discontinuous at the rationals.

5. Let $f \in L^p([a, b])$, $g \in L^q([a, b])$, and $h \in L^r([a, b])$ where $1/p + 1/q + 1/r = 1$. Prove
$$\int_a^b |f(x)g(x)h(x)|\, dx \leq \sqrt[p]{|f(x)|^p\, dx}\, \sqrt[q]{|g(x)|^q\, dx}\, \sqrt[r]{|h(x)|^r\, dx}$$

6. Let $f \in L^p([a, b]) \cap L^q([a, b])$ and let $1 \leq p < r < q < \infty$. Show that $f \in L^r([a, b])$ and show how the L^r-distance from f to 0 can be controlled by the L^p and L^q-distances from f to 0.

7. Two sets are "separated" if
$$[A] \cap B = A \cap [B] = \emptyset$$
   Prove the following:
   a) If A and B are disjoint closed sets in some metric space then they are separated.

b) Prove the same for disjoint open sets.
c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $\rho(p,q) < \delta$, define B similarly with $>$ in place of $<$. Prove that A and B are separated.
d) Prove that every connected metric space with at least two points is uncountable.

8. Let A and B be separated subsets of $\mathbb{R}^k$, suppose $\vec{a} \in A$ and $\vec{b} \in B$, and define

$$\vec{p}(t) = (1-t)\vec{a} + t\vec{b}$$

for $t \in \mathbb{R}$. Define $A_0 = p^{-1}(A)$ and $B_0 = p^{-1}(B)$.
a) Prove that A_0 and B_0 are separated subsets of R^1.
b) Prove that there exists $t_0 \in (0,1)$ such that $\vec{p}(t_0) \in A \cup B$.
c) Prove that every convex subset of $\mathbb{R}^k$ is connected.

9. A metric space is called "separable" if it contains a countable dense subset.
a) Prove that $\mathbb{R}^k$ is separable.
b) Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

10. A collection $\{V_\alpha\}$ of open subsets of X is said to be a "base" for X if the following is true: for every $x \in X$ and every open set $G \subset X$ such that $x \in G$, there is some V_α such that $x \in V_\alpha \subset G$. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.
a) Prove that every separable metric space has a countable base.
b) Prove that every compact metric space X has a countable base.
c) Prove that every compact metric space X is separable.

11. Prove that if X is a complete metric space and $X = \cup_{n=1}^{\infty} F_n$ where each F_n is a closed subset, then at least one F_n has nonempty interior.

12. Do all the problems in Chapter 2 of Kolmogorov and Fomin.

Homework Problems

1. Let X be the space of continuous functions on $[a,b]$ that have k continuous derivatives on $[a,b]$. To measure the distance between two functions and their first k derivatives, one can use the Sobolev metric:

$$\rho_{p,k}(f,g) = \sqrt[p]{\int_a^b |f(x)-g(x)|^p + |f'(x)-g'(x)|^p + \cdots + |f^{(k)}(x) - g^{(k)}(x)|^p \, dx}.$$

where $f^{(k)}$ denotes the kth derivative of f with respect to x.
a) Define $\rho_{\infty,k}$ in the expected manner and show it's a metric.
b) Show that if $1 \le p < \infty$, then $\rho_{p,k}$ is a metric.
c) Give an example of a sequence $\{f_n\}$ that converges to 0 when judged by $\rho_{2,3}$ but diverges to infinity when judged by $\rho_{2,4}$.

2. Let X be the space of pairs (f,g) of continuously differentiable functions on $[0,1]$ that satisfy $f(0) = f(1) = 0$. Define the metric

$$\rho\left((f,g),(F,G)\right) = \sqrt{\frac{1}{2}\int_0^1 (f_x(x) - F_x(x))^2 + (g(x) - G(x))^2 \, dx}$$

where f_x denotes the derivative of f with respect to x.
a) Prove that ρ is a metric on X.
b) Consider the following initial/boundary value problem for the isotropic wave equation on $[0,1]$:

$$\begin{aligned} &q_{tt}(t,x) = q_{xx}(t,x), \quad \forall x \in (0,1), \quad \forall t > 0 \\ &q(0,x) = f(x), \quad q_t(0,x) = g(x), \quad \forall x \in [0,1] \\ &q(t,0) = q(t,1) = 0, \quad \forall t \geq 0. \end{aligned}$$

Show that if $q(t,x)$ is a solution then

$$\rho\left((q(t,\cdot), q_t(t,\cdot)), (0,0)\right) = \rho\left((f,g),(0,0)\right)$$

for all times $t \geq 0$. (The $\cdot$ notation means the following: if $f(x,t) = \cos(tx)$ for $x \in [0,2\pi]$ and $t \in \mathbb{R}$, then $f(\cdot, 8)$ is a function defined on $[0,2\pi]$. It's $\cos(8x)$.
c) Use this to prove that if, given initial data f and g, a solution exists at all then it's the only solution with this initial data. (This is an example of a tailor-made metric. We built the metic to fit the initial/boundary value problem. By being so clever, we proved the uniqueness of solutions very easily!)
d) By the way, why did we assume that all of the functions in X satisfy $f(0) = f(1) = 0$?

3. Let $D \subset R^n$ be a bounded open domain with a smooth boundary. Consider the anisotropic wave equation on D:

$$\begin{aligned} &u_{tt}(t,\vec{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\; u_{x_i,x_j}(t,\vec{x}), \quad \forall \vec{x} \in D, \quad \forall t > 0 \\ &u(0,\vec{x}) = f(\vec{x}), \quad u_t(0,\vec{x}) = g(\vec{x}), \quad \forall \vec{x} \in D \\ &u(t,\vec{x}) = 0, \quad \forall \vec{x} \in \partial D \end{aligned}$$

Let A be the matrix whose (i,j)th entry is a_{ij}. Assume further that A is symmetric and positive definite. Let X be the space of pairs of functions (f,g) that are continuously differentiable on D such that f is zero on ∂D. Find a metric on X such that if $u(t,\vec{x})$ is a solution of the anisotropic wave equation on D then

$$\rho((u(t,\cdot), u_t(t,\cdot)),(0,0)) = \rho((f,g)),(0,0)), \quad \forall t > 0.$$

Use this to show that if the intitial data (f,g) results in a continuously differentiable solution then there is only one such solution. By the way, how did you use the fact that A is positive definite? And why did we assume that f vanishes on ∂D?

4. In example 1 on page 74 of Kolmogorov and Fomin, they use a contraction mapping on the space of continuous functions with the L^∞ metric to prove that there's a solution of

$$f(x) = \phi(x) + \lambda \int_a^b K(x,y) f(y)\, dy$$

if λ is sufficiently small. Show that their mapping A is also a contraction mapping with respect to the L^2 metric (under the appropriate L^2 assumptions on the kernel K and smallness of λ). What about for other $p \in [1,\infty)$?

5. In the real world, one often needs to use a computer to solve the linear algebra problem $A\vec{x} = \vec{b}$. (Try and explain this to the next person you meet at a cocktail party — just kidding!) One never does Gaussian elimination — it's too slow. And if you're going to need to invert the matrix only once, rather than solve a collection of problems with different $\vec{b}$s, or if your matrix is large (1000×1000, say) then one uses iterative methods to solve the problem.

Consider the system:

$$\begin{aligned} 9x_1 + x_2 + x_3 &= b_1 \\ 2x_1 + 10x_2 + 3x_3 &= b_2 \\ 3x_1 + 4x_2 + 11x_3 &= b_3 \end{aligned}$$

The Jacobi method tells you to use your current x to construct your new $\tilde{x}$ by taking the first equation and solving for $\tilde{x}_1$ (assuming the values x_2 and x_3) and so on:

$$\begin{aligned} \tilde{x}_1 &= (b_1 - x_2 - x_3)/9 \\ \tilde{x}_2 &= (b_2 - 2x_1 - 3x_3)/10 \\ \tilde{x}_3 &= (b_3 - 3x_1 - 4x_2)/11 \end{aligned}$$

You then iterate and if your matrix A satisfies the right conditions, the iterates will converge to a solution.

a) Give the Jacobi method for arbitrary linear algebra problems in $\mathbb{R}^n$.

b) Find a condition on A that ensures that the iteration converges.

c) In the Gauss-Seidel method, you use your new guess for $\tilde{x}_1$ in constructing $\tilde{x}_2$ and your new guess for $\tilde{x}_1$ and $\tilde{x}_2$ in constructing $\tilde{x}_3$.

$$\begin{aligned} \tilde{x}_1 &= (b_1 - x_2 - x_3)/9 \\ \tilde{x}_2 &= (b_2 - 2\tilde{x}_1 - 3x_3)/10 \\ \tilde{x}_3 &= (b_3 - 3\tilde{x}_1 - 4\tilde{x}_2)/11 \end{aligned}$$

Repeat questions a) and b) for the Gauss-Seidel method if you can. (Warning – it's harder!)