Mat457Y/Mat1000Y Some practice problems on compact operators and spectral theory

1. Kolmogorov and Fomin, page 251, #2.

2. Kolmogorov and Fomin, page 251, #3.

3. Kolmogorov and Fomin, page 251, #7.

4. Let $E = L^2([0,1])$ and $B \in \mathcal{L}(E,E)$ where B is Hilbert-Schmidt. Show that there is a unique kernel $K \in L^2([0,1] \times [0,1])$ so that

$$(B\phi)(x) = \int_0^1 K(x,y)\phi(y)\, dy$$

for all $\phi \in E$. *Note: if X is compact $L^2(X)$ is the closure of the space of continuous real-valued (or complex-valued) functions on X where the closure is with respect to the L^2 metric. We haven't proven it yet, but $L^2([0,1])$ is a Hilbert space.*

5. Suppose (X,ρ) is a compact metric space (i.e. X is compact) and $S \subset C(X)$ is a subset of the space of continuous real-valued functions on X. The S is called *equicontinuous* if it is "uniformly uniformly continuous". I.e. for all $\epsilon > 0$ there is $\delta > 0$ such that $\rho(x,y) < \delta$ implies $|f(x) - f(y)| < \epsilon$ for all $f \in S$. Assume S is equicontinuous and that there is a constant $B < \infty$ so that $|f(x)| \leq B$ for all $f \in S$ and all $x \in X$. Prove that S has compact closure in $C(X)$.

6. Let $\Omega \subset \mathbb{R}^n$ be open and $X \subset \Omega$ be compact. Let E be the space of bounded continuous real-valued functions on Ω that have a continuous derivative. Let F be the space of continuous functions on X. Let A be the restriction mapping $A : E \Longrightarrow F$ defined by $(Af)(x) = f(x), \forall x \in X$. Prove A is a compact linear operator. *Hint: use previous problem.*

7. Let E be a Hilbert space and let $\{e_j\}$ be an orthonormal basis of E. Assume $A \in \mathcal{L}(E,E)$ is strictly upper triangular. That is, $\langle Ae_j, e_i \rangle = 0$ if $i \geq j$.

   a) If $\dim(E) < \infty$, show that the spectrum of A is $\{0\}$. (Note: Hilbert spaces have countable dimensions by definition, for this part of the problem just assume that E is a complete finite-dimensional real inner product space.)

   b) Give an example to show that if $\dim(E) = \infty$ then we need not have that the spectrum of A is $\{0\}$. *Hint: if $A_n = A|_{span\{e_1, \dots e_n\}}$ then what is $(I - A_n)^{-1}$?*