Mat457Y/Mat1000Y Warm-up exercises and Homework Problems, due September 18, 2002.

Warm-up exercises *If you're writing up solutions for your classmates, you need only write up problems 6, 10, 11, 12, 14, 15, 17, 18, 19, 20, 24, 26.*

1. Prove the triangle inequality for the l^1 metric on $\mathbb{R}^2$.

2. Find the l^p distance between $(0,0)$ and $(1,1)$ as a function of p.

3. What is the largest r for which the l^2 circle of radius r centered at $(0,0)$ fits into the l^1 unit circle? What is the radius of the largest l^1 circle that will fit into the l^2 unit circle?

4. Prove that for $x, y \in \mathbb{R}^2$, $\lim_{p\to\infty} \|x-y\|_p = \|x-y\|_\infty$.

5. Find the shortest distance in the l^1 metric on $\mathbb{R}^2$ from the origin to the line $x_1 + x_2 = 2$. In the l^∞ metric. In the l^2 metric. Is the shortest distance in the l^p metric a monotone function of p?

6. $\mathbb{C}^2$ can be identified in a natural way with $\mathbb{R}^4$: if $z_j = x_j + iy_j$, let $(z_1, z_2) \in \mathbb{C}^2$ correspond to $(x_1, y_1, x_2, y_2) \in \mathbb{R}^4$. Does the unit ball in the l^2 metric on $\mathbb{C}^2$ correspond to the unit ball in the l^2 metric on $\mathbb{R}^4$? Do the l^1 balls correspond to each other?

7. Let $x_j = 1/(j+1)$ for $j = 0, 1, 2, \ldots$. Show that this sequence belongs to $l^p(\mathbb{R}, \mathbb{N})$ if and only if $1 < p \leq \infty$.

8. Let $x_j = 1/(1+ij)$. To which spaces $l^p(\mathbb{C}, \mathbb{Z})$ does this sequence belong?

9. Find a sequence that belongs to $l^p(\mathbb{R}, \mathbb{Z})$ only for $p = \infty$.

10. Let $1 \leq p < \infty$. Is there a sequence $\{x_j\}$ which belongs to $l^r(\mathbb{R}, \mathbb{N})$ for all $r \in (p, \infty]$ and no other r? For all $r \in [p, \infty]$ and no other r?

11. A refinement of the sets $l^p(\mathbb{R}, \mathbb{N})$ is afforded by the following set of sequences:

$$l^{p,k}(\mathbb{R}, \mathbb{N}) = \left\{ x \in l^p(\mathbb{R}, \mathbb{N}) \,\middle|\, \sum_{j=0}^{\infty} \left(1 + j^p + j^{2p} + \cdots + j^{kp}\right) |x_j|^p < \infty \right\}$$

where $1 \leq p < \infty$ and k is a positive integer. Prove or disprove the following:

$$l^{p,k_1}(\mathbb{R}, \mathbb{N}) \subset l^{p,k_2}(\mathbb{R}, \mathbb{N})$$

if $k_1 < k_2$. If it's true, is it a strict inclusion? Can you find (p_1, k_1) and (p_2, k_2) with $p_1 \neq p_2$ and $k_1 \neq k_2$ such that

$$l^{p_1,k_1}(\mathbb{R}, \mathbb{N}) \subset l^{p_2,k_2}(\mathbb{R}, \mathbb{N})?$$

12. The following convergence test, called the Cauchy convergence test, is useful for checking "borderline" series: Let $\{a_n\}$ be a non-increasing sequence of positive numbers. Then $\sum_n a_n$ converges if and only if

$$\sum 2^k a_{2^k}$$

converges. Prove this convergence test is true and then apply it to the following series:

$$\sum \frac{1}{n}, \qquad \sum \frac{1}{n \ln n}, \qquad \sum \frac{1}{n(\ln n)^2}.$$

13. Let X be the set of real 2×2 matrices. For two matrices A and B, define

$$\rho(A, B) = \max\left\{|A_{11} - B_{11}| + |A_{12} - B_{12}|, |A_{21} - B_{21}| + |A_{22} - B_{22}|\right\}.$$

Show this is a metric on X.

14. Find the shortest distance between the origin and the plane $x_1 + x_2 + x_3 = 1$ in the l^1, l^2, and l^∞ metrics on $\mathbb{R}^3$.

15. In $\mathbb{R}^n$, find the shortest distance between the origin and the plane $x_1 + x_2 + \cdots + x_n = 1$ in the l^1, l^2, and l^∞ metrics.

16. Let $0 < p < 1$. Show that

$$\rho(x, y) = \sum_{j=1}^{n} |x_j - y_j|^p$$

defines a metric on $\mathbb{R}^n$.

17. Prove that if ρ and ρ' are equivalent metrics on X then a sequence x_n converges to x_∞ in the metric ρ if and only if it converges in the metric ρ'.

18. Prove that for $1 \le p \le \infty$ the l^p metrics on $\mathbb{R}^n$ are mutually equivalent. (I.e., prove that l^p is equivalent to l^q for any p and q in $[1, \infty]$.

19. Fix $\epsilon > 0$. Find the largest $\delta(\epsilon, n)$ such that for $x \in \mathbb{R}^n$ if $\|x\|_2 < \delta$ then $\|x\|_1 < \epsilon$. What happens to $\delta(\epsilon, n)$ as $n \to \infty$? What is the largest $\delta'(\epsilon, n)$ such that $\|x\|_1 < \delta'$ guarantees $\|x\|_2 < \epsilon$? What happens as $n \to \infty$? In the limit $n \to \infty$, the *sets* $l^1(\mathbb{R}, \mathbb{N})$ and $l^2(\mathbb{R}, \mathbb{N})$ are certainly different. But are the metrics equivalent when used on sequences that belong to $l^1(\mathbb{R}, \mathbb{N})$?

20. Find, if possible, a sequence x_n in $l^1(\mathbb{R}, \mathbb{N})$ that converges to 0 in the l^1 metric but not in the l^∞ metric. Find, if possible, a sequence x_n in $l^1(\mathbb{R}, \mathbb{N})$ that converges to 0 in the l^∞ metric but not in the l^1 metric. What does this mean about the equivalence of metrics?

21. Same as the preceding exercise, but replace l^∞ by l^2.

22. Let $1 \le p \le 2$ and pick $x \in l^p(\mathbb{R}, \mathbb{N})$. Prove that the sequence $x_1^2, x_2^2, x_3^2, \ldots$ belongs to $l^1(\mathbb{R}, \mathbb{N})$. Is this conclusion still true if $2 < p < \infty$?

23. Let $x \in l^q(\mathbb{R}, \mathbb{N})$. Show that the sequence $|x_1|^{q-1}, |x_2|^{q-1}, |x_2|^{q-1}, \ldots$ belongs to $l^p(\mathbb{R}, \mathbb{N})$ where $1/p + 1/q = 1$.

24. Let $x \in l^2(\mathbb{R}, \mathbb{N})$. Show that the sequence $\{x_j/j^\alpha\}$ lies in $l^1(\mathbb{R}, \mathbb{N})$ for $\alpha > 1/2$ and find an example of a sequence for which $\{x_j/\sqrt{j}\}$ does not lie in $l^1(\mathbb{R}, \mathbb{N})$.

25. Let $x_1, x_2, \ldots x_n > 0$. Show that

$$\left(\sum_{j=1}^{n} x_j\right)\left(\sum_{j=1}^{n} \frac{1}{x_j}\right) \ge n^2$$

26. Let f and g be continuous functions on $[a, b]$. Define

$$\rho(f, g) = \left(\int_a^b |f(x) - g(x)|^p \, dx\right)^{1/p}$$

Show that if $0 < p < 1$ then ρ is not a metric.

Homework Problems

1. Let (X, ρ) be a metric space. Show that for all $x, y, z \in X$, one has $|\rho(x, z) - \rho(y, z)| \le \rho(x, y)$. Draw a picture to illustrate the meaning of this inequality. Show that for all $x, y, z, u \in X$, one has $|\rho(x, z) - \rho(y, u)| \le \rho(x, y) + \rho(z, u)$. Draw a picture to illustrate the meaning of this inequality.

2. Verify that

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 = \sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}(a_j b_k - a_k b_j)^2.$$

Deduce the Cauchy-Schwartz inequality from this identity.

3. Let f and g be continuous functions on $[a,b]$. Let $1 < p, q < \infty$. Prove Hölder's integral inequality:

$$\int_a^b f(x)g(x)\,dx \le \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\,dx\right)^{\frac{1}{q}}.$$

Let $1 \le p < \infty$. Prove Minkowski's integral inequality:

$$\left(\int_a^b |f(x)+g(x)|^p\right)^{1/p} \le \left(\int_a^b |f(x)|^p\right)^{1/p} + \left(\int_a^b |g(x)|^p\right)^{1/p}.$$

Prove that these inequalities hold for Riemann integrable functions as well.

4. Fill in the following outline for a different proof of Young's inequality:

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

for $a, b > 0$, with $p, q \in (1, \infty)$ and $1/p + 1/q = 1$. First, it suffices to prove that for $a, b > 0$ and $0 < \eta < 1$ that

$$a^\eta b^{1-\eta} \le \eta a + (1-\eta)b. \tag{1}$$

This follows from

$$a^\eta - \eta a - (1-\eta) \le 0.$$

Prove this last inequality and show how it implies the second inequality and show how that inequality implies Young's inequality. Finally, show how this proof shows that

$$ab = \frac{a^p}{p} + \frac{b^q}{q}$$

if and only if $b = a^{p-1}$.

5. Let $a_1, a_2, \ldots a_n$ be positive numbers. The usual average

$$\frac{a_1 + a_2 + \ldots a_n}{n}$$

is the arithmetic mean and

$$\sqrt[n]{a_1 a_2 \ldots a_n}$$

is the geometric mean. The "theorem of the arithmetic and geometric means" says

$$\sqrt[n]{a_1 a_2 \ldots a_n} \le \frac{a_1 + a_2 + \ldots a_n}{n}$$

with equality holding only if $a_1 = a_2 = \cdots = a_n$. If $n = 2$ then the above inequality is the same as the inequality (1) with $\eta = 1/2$. Prove the following generalization of the theorem of the arithmetic and geometric means:

$$a_1^{\eta_1} \ldots a_n^{\eta_n} \le \eta_1 a_1 + \ldots \eta_n a_n$$

where $\eta_1 + \cdots + \eta_n = 1$, with equality holding only if $a_1 = \cdots = a_n$.