A Non-Reflexive Banach Space Isometric with Its Second Conjugate Space

Robert C. James

Proceedings of the National Academy of Sciences of the United States of America,

Stable URL:
http://links.jstor.org/sici?sici=0027-8424%2819510315%2937%3A3C174%3AANBSIW%3E2.0.CO%3B2-I

Your use of the JSTOR archive indicates your acceptance of JSTOR’s Terms and Conditions of Use, available at http://www.jstor.org/about/terms.html. JSTOR’s Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

Proceedings of the National Academy of Sciences of the United States of America is published by National Academy of Sciences. Please contact the publisher for further permissions regarding the use of this work. Publisher contact information may be obtained at http://www.jstor.org/journals/nas.html.

Proceedings of the National Academy of Sciences of the United States of America
©1951 National Academy of Sciences

JSTOR and the JSTOR logo are trademarks of JSTOR, and are Registered in the U.S. Patent and Trademark Office. For more information on JSTOR contact jstor-info@umich.edu.

©2002 JSTOR
A NON-REFLEXIVE BANACH SPACE ISOMETRIC WITH ITS SECOND CONJUGATE SPACE

BY ROBERT C. JAMES

UNIVERSITY OF CALIFORNIA

Communicated by J. von Neumann, December 5, 1950

A Banach space $B$ is isometric with a subspace of its second conjugate space $B^{**}$ under the “natural mapping” for which the element of $B^{**}$ which corresponds to the element $x_0$ of $B$ is the linear functional $F_{x_0}$ defined by $F_{x_0}(f) = f(x_0)$ for each $f$ of $B^*$. If every $F$ of $B^{**}$ is of this form, then $B$ is said to be reflexive and $B$ is isometric with $B^{**}$ under this natural mapping. The purpose of this note is to show that $B$ can be isometric with $B^{**}$ without being reflexive. The example given to show this is a space isomorphic with a Banach space known to not be reflexive, but to be isomorphic with its second conjugate space.\(^1\)

A sequence ${\{x^n\}}$ of elements of a Banach space $B$ is said to be a basis for $B$ if for each $x$ of $B$ there is a unique sequence of numbers $\{a_n\}$ such that $x = \sum_{1}^{\infty} a_n x^n$ in the sense that $\lim_{n \to \infty} \|x - \sum_{1}^{n} a_n x^n\| = 0$. A fundamental sequence $\{x^n\}$ is a basis if and only if there is a positive number $\epsilon$ such that $\|\sum_{1}^{n+p} a_n x^n\| \geq \epsilon \|\sum_{1}^{n} a_n x^n\|$ for any positive integers $n$ and $p$ and numbers $\{a_i\}$.\(^2\) If $\epsilon = 1$, the basis will be called an orthogonal basis. But for any basis $\{x^n\}$, $\|x\| = \sup_n \|\sum_{1}^{n} a_n x^n\|$ for $x = \sum_{1}^{\infty} a_n x^n$ defines a new norm $\|\|$ which is equivalent to $\|\|$ and for which $\{x^n\}$ is an orthogonal basis.\(^3\) Hence if $B$ has a basis $\{x^n\}$ for which $\lim_{n \to \infty} \|f\|_n = 0$ for each $f$ of $B^*$, where $\|f\|_n$ is the norm of $f$ on $x^{n+1} \oplus x^{n+2} \oplus \ldots$, then the following theorem describes $B^{**}$ completely if the basis is orthogonal and describes $B^{**}$ to within an isomorphism if the basis is not an orthogonal basis.

**Theorem.** Let $B$ be a Banach space with an orthogonal basis $\{z^n\}$ for which $\lim_{n \to \infty} \|f\|_n = 0$ for each $f$ of $B^*$, where $\|f\|_n$ is the norm of $f$ on $z^{n+1} \oplus z^{n+2} \oplus \ldots$. Then $\{g^n\}$ is a basis for $B^*$ if $g^n(z^m) = \delta_{mn}^n$ for each $n$ and $m$. If $F \in B^{**}$, then $\|F\| = \lim_{n \to \infty} \|\sum_{1}^{n} F_i\|$ where $F_i = F(g^i)$. If the sequence $\{F_n\}$ is such that $\lim_{n \to \infty} \|\sum_{1}^{n} F_i\| < + \infty$, then $F \in B^{**}$ if one defines $F(f) = \sum_{1}^{\infty} F_i f_i$ for each $f = \sum f_i g_i$ of $B^*$.

**Proof:** It has been previously known that $\{g^n\}$ is a basis for $B^*$.\(^4\) It
follows from this that \( F(f) = \sum_{i=1}^{\infty} F_{f_i} \) for each \( F \) of \( B^{**} \) and each \( f = \sum_{i=1}^{\infty} f_i g_i \) of \( B^* \), where \( F_i = F(g_i) \). But, for each \( f = \sum_{i=1}^{\infty} f_i g_i \), \( \sum_{i=1}^{n} F_{f_i} = |f(\sum_{i=1}^{n} F_i g_i)| \leq \|f\|\|\sum_{i=1}^{n} F_i g_i\| \). Thus \( \|\sum_{i=1}^{n} F_{f_i}\| \leq \|f\| (\lim_{n \to \infty} \|\sum_{i=1}^{n} F_i g_i\|) \), and \( \|F\| \leq \lim_{n \to \infty} \|\sum_{i=1}^{n} F_i g_i\| \). For a fixed \( n \), let \( u^n = \sum_{i=1}^{n} F_i g_i \). Define a linear functional \( h \) by \( h(g_i) = 0 \) for \( i > n \) and \( h(u^n) = \|u^n\| \). Then \( |h(au^n + \sum_{i=n+1}^{\infty} a_i g_i)| = \|au^n\| \leq \|au^n + \sum_{i=n+1}^{\infty} a_i g_i\| \). Thus \( \|h\| = 1 \) on \( u^n \oplus z^{n+1} \oplus z^{n+2} \oplus \ldots \).

Extend \( h \) to all of \( B \) so that \( \|h\| = 1 \) on \( B \). Then, for this \( h \), \( h = \sum_{i=1}^{\infty} h_i g_i \) with \( h_i = 0 \) for \( i > n \), so that \( |\sum_{i=1}^{n} F_{h_i}| = \|\sum_{i=1}^{n} F_{h_i}\| = \|h(u^n)\| = \|u^n\| \leq \|F\| \).

Since this can be done for each \( n \), it follows that \( \|F\| \geq \|\sum_{i=1}^{n} F_i g_i\| \) for each \( n \) and \( \|F\| \geq \lim_{n \to \infty} \|\sum_{i=1}^{n} F_i g_i\| \). It has thus been shown that \( \lim_{n \to \infty} \|\sum_{i=1}^{n} F_i g_i\| = \|F\| \) for each element \( F = \{F_n\} \) of \( B^{**} \). Now suppose that \( \{F_n\} \) is a sequence such that \( \lim_{n \to \infty} \|\sum_{i=1}^{n} F_i g_i\| = M < + \infty \). Then \( \|\sum_{i=1}^{n+1} F_i g_i\| \leq 2M \).

Thus for any fixed \( f \in B^* \), \( \sum_{i=1}^{n+1} F_{f_i} = |f(\sum_{i=1}^{n+1} F_i g_i)| \leq \|f\|_n(2M) \), so that it follows from \( \lim_{n \to \infty} \|f\|_n = 0 \) that \( \sum_{i=1}^{\infty} F_{f_i} \) is convergent. Thus \( F(f) = \sum_{i=1}^{\infty} F_{f_i} \) is defined for each \( f \in B^* \) and \( \|F\| = \lim_{n \to \infty} \|\sum_{i=1}^{n} F_i g_i\| \).

**Example:** For \( x = (x_1, x_2, x_3, \ldots) \), let

\[
\|x\| = \text{l. u. b.} \left[ \sum_{i=1}^{n} (x_{p_i} - x_{p_{i+1}})^2 + (x_{p_{n+1}} - x_{p_1})^2 \right]^{1/2},
\]

(1)

where the l. u. b. is over all positive integers \( n \) and all finite increasing sequences of at least two positive integers \( p_1, p_2, \ldots, p_{n+1} \). Let \( B \) be the Banach space of all \( x \) for which \( \|x\| \) is finite and \( \lim_{n \to \infty} x_n = 0 \). Then \( B \) is isometric with \( B^{**} \), but is isometric under the natural mapping with a closed maximal linear subspace of \( B^{**} \).

**Proof:** For \( x = (x_1, x_2, \ldots) \), let

\[
\|\|x\|\| = \text{l. u. b.} \left[ \sum_{i=1}^{n} (x_{p_{2i-1}} - x_{p_{2i}})^2 + (x_{p_{2n+1}} - x_{p_{2n+2}})^2 \right]^{1/2},
\]

(2)

where the l. u. b. is over all positive integers \( n \) and finite increasing se-
quences of positive integers \( p_1, p_2, \ldots, p_{2n+1} \). It follows from \( \lim_{n \to \infty} x_n = 0 \) and \( ||x|| = |x_p| \) that \( ||x|| = |x_p| \) for each \( p \). Clearly \( ||x|| = |x_p| \) for each \( p \). But by grouping alternating terms of (1) and isolating \( x_{p_1} \), one gets \( ||x|| \leq 1 \) if \( b \). \( \{x_{p_1} + (x_{p_{n+1}} - x_{p_2})^2 + (x_{p_{n+2}} - x_{p_2})^2 + \ldots\}^{1/2} + (x_{p_n} - x_{p_{n+1}})^2 + (x_{p_{n-2}} - x_{p_{n-1}})^2 + \ldots\}^{1/2} \leq 3||x||. \) But extra terms can be introduced in (2) to give a sum of type (1), except for replacing \( (x_{p_{2n+1}} - x_{p_1}) \) by \( (x_{p_{2n+1}} - x_{p_1}) \). Thus \( ||x|| \leq 2||x||. \) Since \( 1/2||x|| \) \( \leq ||x|| \leq 3||x||, \) these two norms are equivalent. But the Banach space of all \( x = (x_1, x_2, \ldots) \) for which \( \lim_{n \to \infty} x_n = 0 \) and \( ||x|| \) is finite is known to not be reflexive, but to be isometric under the natural mapping with a closed maximal linear subspace of its second conjugate space. \(^1\) Hence this is also true of the space \( B \).

Let \( z^n = (0, 0, \ldots, 0, 1, 0, \ldots) \) be the element of \( B \) whose components are all zero except for the \( n \)th component, which is 1. Then \( z^1 = z^2 = \ldots = B, \) so that \( \{z^n\} \) is an orthogonal basis for \( B \) if \( ||\sum_{1}^{n} a_i z^i + \sum_{n+1}^{n+p} b_i z^i || \geq ||\sum_{1}^{n} a_i z^i || \) for all numbers \( \{a_i\} \) and \( \{b_i\} \) and positive integers \( n \) and \( p. \) Since \( \sum_{1}^{n} a_i z^i \) has only a finite number of non-zero components, a sequence \( p_1, p_2, \ldots, p_{k+1} \) can be chosen so that

\[
||\sum_{1}^{n} a_i z^i || = \left[ \sum_{i=1}^{k} (a_{p_i} - a_{p_{i+1}})^2 + (a_{p_{k+1}} - a_{p_1})^2 \right]^{1/2}, \tag{3}
\]

where \( a_r = 0 \) if \( r > n. \) If \( p_{k+1} \leq n, \) then it is immediate from (1) and (3) that \( ||\sum_{1}^{n} a_i z^i + \sum_{n+1}^{n+p} b_i z^i || \geq ||\sum_{1}^{n} a_i z^i ||. \) If \( p_{k+1} > n, \) then each \( p_i \) with \( p_i > n \) can be replaced by some \( p_i > n + p \) without changing the value of (3), since \( a_r = 0 \) if \( r > n. \) But it will then again follow from (1) and (3) that \( ||\sum_{1}^{n} a_i z^i + \sum_{n+1}^{n+p} b_i z^i || \geq ||\sum_{1}^{n} a_i z^i ||. \) For \( B \) with the norm \( ||\cdot|| \) and hence also for \( B \) with the norm \( ||\cdot|| \), it is known that \( \lim_{n \to \infty} ||f|| \leq 0, \) where \( ||f|| \) is the norm of \( f \) on \( z^{n+1} = z^{n+2} = \ldots. \) \(^1\) Hence by Theorem 1 above, \( B^{**} \) is the space of all \( F = (F_1, F_2, \ldots) \) for which \( ||F|| = \lim_{n \to \infty} ||\sum_{1}^{n} F_i z^i || \) is finite. Thus for \( F \) to belong to \( B^{**}, \) it is necessary that \( \lim_{n \to \infty} F_n \) exist. Consider the correspondence:

\[
x = (x_1, x_2, \ldots) \leftrightarrow (x_2 - x_1, x_3 - x_1, \ldots, x_n - x_1, \ldots) = (F_1, F_2, \ldots) = F_x.
\]
To show that \( \|x\| = \|F_x\| \), first consider a sum \( \sum_{i=1}^{n} (x_{p_i} - x_{p_{i+1}})^2 + (x_{p_{n+1}} - x_{p_1})^2 \). If \( p_1 \geq 2 \), this is equal to \( \sum_{i=1}^{n} (F_{p_i} - F_{p_{i+1}})^2 + (F_{p_{n+1}} - F_{p_1})^2 \). If \( p_1 = 1 \), it is equal to \( \sum_{i=2}^{n} (F_{p_{i-1}} - F_{p_{i+1}})^2 + (F_{p_{n+1}} - F_{N})^2 + (F_{N} - F_{p_2})^2 \), if \( N > p_{n+1} - 1 \) and \( F_N \) is replaced by zero. Since \( \|\sum_{i=1}^{n} F_{p_i}^i\| \) is a monotonically increasing function of \( n \), it follows that \( \|x\| \leq \|F_x\| \), where \( \|F_x\| = \lim_{n \to \infty} \|\sum_{i=1}^{n} F_{p_i}^i\| \). Now consider a sum \( \sum_{i=1}^{n} (F_{p_i} - F_{p_{i+1}})^2 + (F_{p_{n+1}} - F_{p_1})^2 \), formed for the element \( \sum_{i=1}^{m} F_{i}^i \), where \( F_p \) is to be replaced by 0 if \( p > m \). If \( p_{n+1} \leq m \), then this sum is equal to \( \sum_{i=1}^{n} (x_{p_{i+1}} - x_{p_{i+1}} + 1)^2 + (x_{p_{n+1} + 1} - x_{p_{n+1}})^2 \). Now suppose that \( p_{k+1} > m \), but \( p_i \leq m \) if \( i \leq k \). Then the sum becomes \( \sum_{i=1}^{k-1} (F_{p_i} - F_{p_{i+1}})^2 + (F_{p_k})^2 + (F_{p_1})^2 \) \( \sum_{i=1}^{k-1} (x_{p_{i+1}} - x_{p_{i+1}} + 1)^2 + (x_{p_k} + 1 - x_1)^2 \) \( + (x_1 - x_{p_{k+1}})^2 \). Thus \( \|x\| \geq \|\sum_{i=1}^{n} F_{p_i}^i\| \) for each \( n \). Hence \( \|x\| = \|F_x\| \) and \( x \leftrightarrow F_x \) is an isometry with domain equal to \( B \). But if \( F = (F_1, F_2, \ldots) \) is an element of \( B^{**} \), and \( \lim_{n \to \infty} F_n = L \), then \( x_F = (-L, F_1 - L, F_2 - L, \ldots) \) is, by the above, an element of \( B \) for which \( \|x_F\| = \|F\| \) and \( x_F \leftrightarrow F \). Thus the range of the isometry is \( B^{**} \).

4 James, *loc. cit.*, Theorem 3.