Compactness

Heine-Borel theorem: Any cover of the closed interval $[a,b]$ by a system of open intervals (or open sets) has a finite subcover.

i.e. $[a,b] = \bigcup_\alpha U_\alpha \Rightarrow [a,b] = \bigcup_1^n U_i$ some n

defn: (X, τ) is compact if every open cover of X has a finite subcover. A compact Hausdorff space is called a compactum

Note: It's every cover. E.g.

$(0,1]$ is not compact because the open cover $\{(\frac{1}{n}, \frac{2}{n}) \mid n \in \mathbb{N}\}$ has no finite subcover

defn: A system of subsets $\{A_\alpha\}$ of a set X is centered if every finite intersection

$$\bigcap_1^n A_i \neq \emptyset$$

<u>theorem</u>: $(X, \mathcal{T})$ is compact if and only if every centered system of closed subsets has nonempty intersection

proof:

($\Longrightarrow$) Assume not. Then $\exists$ centered system of closed subsets w/ empty intersection.

i.e. $\bigcap_1^n F_j \neq \phi$ for each finite subcollection

but $\bigcap_\alpha F_\alpha = \phi$.

Let $G_\alpha = X - F_\alpha$. G_α is open and since $\bigcap_\alpha F_\alpha = \phi$, we know $\bigcup_\alpha G_\alpha = X$. $\Rightarrow \{G_\alpha\}$ is an open cover w/ no finite subcover $\Rightarrow X$ is not compact.

($\Longleftarrow$) Let $\{G_\alpha\}$ be an open cover of X. We want to find a finite subcover.

Since $\bigcup_\alpha G_\alpha = X$, if $F_\alpha = X - G_\alpha$ then $\bigcap_\alpha F_\alpha = \phi$.

But this means that $\{F_\alpha\}$ cannot be a centered system of closed subsets

$\Rightarrow \exists$ finite subcollection $\ni$ $\bigcap_1^n F_i = \phi$

$\Rightarrow$ this subcollection also covers X. //

<u>theorem</u>: A closed subset of a compact topological space is itself compact

<u>proof 1</u>:

Let $\{V_\alpha\}$ cover $A \subset X$. Since A is closed, $\{V_\alpha\} \cup \{X-A\}$ is an open cover of X. $\Rightarrow$ $\exists$ finite subcover that covers X. This subcover also covers A. $\Rightarrow$ A is compact.

I really should be careful, we talk of <u>spaces</u> being compact. So what I really mean above is (A, τ_A) is a compact topological space if A is closed in X and (X, τ) is a compact topological space.

It still works since $V \in \tau_A \Rightarrow V = U \cap A$ some $U \in \tau$. So if $\{V_\alpha\}$ is a collection of τ_A open sets that covers A then $\{U_\alpha\} \cup \{X-A\}$ is a collection of τ-open sets that cover X. The finite subcollection $\{U_i\}$ then induces the finite subcollection $\{V_i\}$ of τ_A open subsets.

<u>theorem</u>: let K be a compact subset of a Hausdorff space T. Then K is closed in T

<u>proof</u>: Take $y \in T - K$. Given $x \in K$ $\exists$ U_x and V_y $\ni$ $U_x \cap V_y = \emptyset$ U_x, U_y open. (since T is Hausdorff)

$\{U_x\}$ covers $K \Rightarrow \exists$ finite subcover.

$K \subset \bigcup_1^n U_{x_i}$ let $V = \bigcap_1^n V_{x_i}$

$\Rightarrow V \cap K = \emptyset \Rightarrow y \notin [K] \Rightarrow K$ is closed. //

<u>theorem</u>: If K is a compactum then it is T_4.

<u>proof</u>: let $X, Y \subset K$ be disjoint & closed given $y \in Y$ $\exists$ open set containing X (O_x) and an open set containing y (U_y). (Why? we know X is a compactum bec. it's closed and K is a compactum. And we repeat the construction from the prev. theorem...) We can cover Y w/ such U_y. But finitely many of them will cover $Y \Rightarrow Y \subset \bigcup_1^n U_{y_i}$ and $X \subset \bigcap_1^n O_{x_i}$ and they're disjoint by constr. //

theorem: if X is a compact space and $f: X \to Y$ is a continuous mapping onto topological space Y then $f(X) = Y$ is compact.

proof: Let $\{V_\alpha\}$ cover Y then $f^{-1}(V_\alpha)$ is an open cover of X. $\Rightarrow$ only need finitely many to cover $X \Rightarrow$ only need finitely many of $\{V_\alpha\}$ to cover Y. //

theorem: If T is a compact space then any infinite subset of T has at least one limit point

proof: If $\exists$ infinite set $X \subset T$ w/ no limit point then $\exists$ countable subset of X w/ no limit point. (Why? If $\{x_n\}$ has limit point $t \in X$ then every neighborhood of t has infinitely many elts of the sequence. But then t is a limit point of X too!).

$\{x_1, x_2, x_3, \ldots\}$
$\{x_2, x_3, x_4, \ldots\}$
$\vdots$
$\{x_n, x_{n+1}, \ldots\}$

form a centered system of closed subsets of T w/ empty intersection. $\times$

An alternate proof of the theorem is as follows:

If no $t \in T$ is a limit point of X, then for each t $\exists$ open set U_t such that U_t contains t and contains only one point of X.

Then $\{U_t\}$ covers T by construction. And $\nexists$ finite subcover because a finite subcover would miss members of X (because X is finite.)

Theorem: if $E \subset \mathbb{R}^k$ has one of the following three properties then it has the other two:

(a) E is closed and bounded

(b) E is compact

(c) Every infinite subset of E has a limit point in E

(here, we assume the metric topology on $\mathbb{R}^k$ with $\rho(x,y) = |\Sigma |x_i - y_i|$ for example.)

Is there an analogous theorem ~~for~~ for topological spaces?

<u>theorem</u>: If (X,τ) is a topological space, then the following are equivalent:

(a) X is compact

(b) every net in X has a cluster point

(c) every net in X has a convergent subnet.

this is the topological vector space version of the Bolzano-Weierstrauss theorem

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(X,τ) is <u>countably compact</u> if every countable open cover of X has a finite subcover

(X,τ) is <u>sequentially compact</u> if every sequence in X has a convergent subsequence.

compact $\Rightarrow$ countably compact

In metric spaces, compact $\Leftrightarrow$ sequentially compact

and

sequentially compact $\Rightarrow$ countably compact