Separation Axioms

<u>T_0</u>: if $x \neq y$ then either $\exists\, U_x$ such that $y \notin U_x$ or $\exists\, U_y$ such that $x \notin U_y$ (note: throughout, U_x will mean "an open set that contains x")

<u>T_1</u>: if $x \neq y$ then $\exists\, U_x$ and U_y such that $x \notin U_y$ and $y \notin U_x$

<u>T_2</u>: if $x \neq y$ then $\exists$ <u>disjoint</u> U_x and U_y (also called "Hausdorff")

<u>T_3</u>: X is T_1 <u>and</u> given A closed and $x \in A^c$ then $\exists\, U_x$ and V (open) such that $x \in U_x$, $A \subset V$, and $U_x \cap V = \emptyset$ (also called "regular")

<u>T_4</u>: X is T_1 <u>and</u> given A and B closed and disjoint, $\exists\, U$ and V open and disjoint such that $A \subset U$ and $B \subset V$. (also called "normal")

A space that's not even T_0?

(X, τ) where $\tau = \{\emptyset \text{ and } X\}$

Notes: if $x_n \to x$ for some $x \in X$ then the sequence converges to y for all $y \in X$. Even worse... all infinite sequences are convergent!

A space that's T_0 but not T_1?

(X, τ) where $X = \{a, b\}$

and $\tau = \{\emptyset, X, \{a\}\}$

then $\nexists$ open set that contains b but doesn't contain a.

Note: if $x_n \to b$ then $x_n \to a$ too.

But $x_n \to a \Rightarrow x_n \not\to b$.

A space that's T_1 but not T_2?

<u>ex 1:</u> $\mathbb{R}$ w/ co-countable topology.

$\tau = \emptyset \cup \{$sets w/ finite or countable complement$\}$

then all open sets intersect each other
$\Rightarrow T_2$ is impossible. And (X, τ) is
T_1 since $U_x = \mathbb{R} - \{y\}$ and
$U_y = \mathbb{R} - \{x\}$ work.

<u>ex 2:</u> $\mathbb{R}$ w/ co-finite topology

$\tau = \emptyset \cup \{$sets w/ finite complement$\}$

(X, τ) is T_1 but not T_2 (see above).

<u>Note:</u> $\mathbb{R}$ w/ co-countable topology has
$x_n \to x \Rightarrow \{x_n\}$ is eventually stationary
$\Rightarrow$ limits are unique

$\mathbb{R}$ w/ co-finite topology has that
any sequence $\{x_n\}$ w/ infinitely many distinct
points will converge to $x \in \mathbb{R}$ for
<u>any</u> x. limits definitely not unique!

<u>theorem</u>: if (X, τ) is T_2 then every convergent sequence has a unique limit

<u>theorem</u>: if (X, τ) is first countable and every convergent sequence has a unique limit then (X, τ) is T_2

A space that's T_2 but not T_3?

$X = \mathbb{R}$ and the open sets are of the form $U \cup (V \cap \mathbb{Q})$ where U and V are open in the usual metric topology $(\rho(x,y) = |x-y|)$.

<u>theorem</u>: all metric spaces are T_4.

<u>theorem</u>: X is T_1 if and only if every set consisting of a single point is closed.

<u>proof</u>:

($\Leftarrow$) Assume $\{x\}$ is closed. then $X - \{x\}$ is open

$\Rightarrow$ given $x \neq y$ take $U_x = X - \{y\}$ and $U_y = X - \{x\}$. $\Rightarrow$ (X, τ) is T_1.

($\Leftarrow$) assume (X, τ) is T_1. I want to show $\{x\}$ is closed $\Rightarrow$ I want to show $X - \{x\}$ is open. $\Rightarrow$ I want to show that given $y \in X - \{x\}$ $\exists\, U_y \subset X - \{x\}$. this follows directly from defn of T_1. //

Corollary:

$T_4 \Rightarrow T_3 \Rightarrow T_2$

Let's go back to the co-finite and co-countable topologies on $\mathbb{R}$

($\mathbb{R}$, co-countable topology)

is T_1, is not T_2

is not first countable (doesn't have countable local base)

convergent sequences are eventually stationary

if $f: (\mathbb{R}, \text{co-countable}) \to (\mathbb{R}, \text{usual metric})$ is continuous then f is the constant function

($\mathbb{R}$, co-finite topology)

is T_1 not T_2

is compact (let $\{U\}$ be a cover. Take any U from the cover and call it U_1. then $X\ \mathbb{R} - U_1$ is finite $\Rightarrow$ take a finite collection of $\{U\}$ to cover $\mathbb{R} - U_1$ ... done!)

convergent sequences converge to all points

if $f: (\mathbb{R}, \text{co-finite}) \to (\mathbb{R}, \text{metric})$ is continuous then f is constant

If $f: (\mathbb{R}, \text{cofinite}) \to (\mathbb{R}, \text{cofinite})$ is continuous then and nonconstant then $f^{-1}(\{y\})$ is a finite collection of points for each $y \in \mathbb{R}$

($\mathbb{R}$, cofinite) is the 1-d version of the Zariski topology ... important in algebraic geometry!!

Urysohn's Metrization Theorem

if (X, τ) is T_4 and second-countable, then (X, τ) is metrizable.

i.e. $\exists$ a metric on X so that the open sets induced by the metric are precisely those in τ.

Note: Metrizable $\Rightarrow T_4$ but metrizable $\not\Rightarrow$ second countable. Think of $\ell^\infty(\mathbb{R}, \mathbb{N})$.

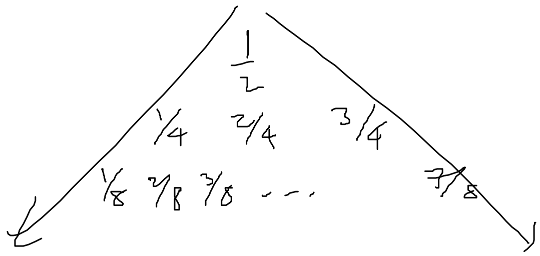
Lemma: Suppose A and B are disjoint closed sets in a T_4 space. Let

$$\Delta = \left\{ \frac{k}{2^n} \;\middle|\; n \geq 1 \text{ and } 0 < k < 2^n \right\}$$

be the set of diadic rational numbers in $(0,1)$. Then $\exists$ a family of open sets $\{U_r \mid r \in \Delta\}$ such that

1) $A \subset U_r \subset B^c$ for each $r \in \Delta$
2) $r < s \Rightarrow \overline{U_r} \subset U_s$

Δ is



Imagine $A = (-\infty, 3]$ $B = [6, \infty)$

then we can fill the space between A and B easily:

$$U_r = (-\infty, a + r(b-a)) \qquad r \in \Delta$$

then $r < s \Rightarrow \overline{U_r} \subset U_s$ and

$A \subset U_r \subset B^c \quad \forall\, r \in \Delta$

Note 1: we didn't need $r \in \Delta$ here. $r \in (0,1)$ would've worked just as well

Note 2: $U_r = \left(-\infty, a + r\left(\frac{b-a}{2}\right)\right)$

would've also met the requirements of the lemma, even though

$$\bigcup_r U_r \neq B^c$$

(there's left over space). The theorem doesn't say $\bigcup U_r = B^c$. just $\bigcup_r U_r \subset B^c$.

Proof of theorem:

$A \cap B = \emptyset$ and A, B closed

$\Rightarrow \exists$ open sets U and V such that

$A \subset U$, $B \subset V$, $U \cap V = \emptyset$.

Let $U_{1/2} = U$. then $A \subset U_{1/2} \subset B^c$.

Now...

$A \cap U_{1/2}^c = \emptyset$ and $U_{1/2}^c$ is a closed set.

$\Rightarrow \exists$ U and V open so that

$A \subset U$ and $U_{1/2}^c \subset V$ and $U \cap V = \emptyset$

Let $U_{1/4} = U$. then $A \subset U_{1/4} \subset U_{1/2}$ by construction. Also, $\overline{U_{1/4}} \subset U_{1/2}$

Also,

$\overline{U_{1/2}} \cap B = \emptyset$

$\Rightarrow \exists$ U and V open so that

$\overline{U_{1/2}} \subset U$ and $B \subset V$ and $U \cap V = \emptyset$

Let $U_{3/4} = U$. Then $U_{1/2} \subset U_{3/4} \subset B^c$ by construction. And $\overline{U_{1/2}} \subset U_{3/4}$.

So we've constructed

$A \subset U_{1/4} \subset U_{1/2} \subset U_{3/4} \subset B^c$ so that

$\overline{U_{1/4}} \subset U_{1/2}$, $\overline{U_{1/2}} \subset U_{3/4}$

Continuing in this way, we construct a nested family of open sets such that $r < s \Rightarrow \overline{U_r} \subset U_s$ for all $r \in \Delta$.

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By construction

$$A \subset \bigcap_{\Delta} U_r$$

$$\bigcup_{\Delta} U_r \subset B^c$$