A collection of sets is a <u>local base</u> for the topology at $x \in X$ if

1) $x \in G$ for each $G \in \{G_x\}$
2) given any open set U with $x \in U$ $\exists$ some $G \in \{G_x\}$ with $G \subset U$.

<u>definition</u> if (X, τ) has a countable base then the topological space satisfies the second axiom of countability

<u>exs</u> $\mathbb{R}^n$ with d^p metric $1 \leq p \leq \infty$ satisfies the second axiom of countability because we can take a dense subset (the points with rational coordinates --- a countable set) and at each point, we consider the spheres of rational radius. This give a countable base for the topology.

We built the countable base on the back of a countable dense set in that example

<u>theorem:</u> If $(X, \mathcal{T})$ has a countable base then X has a countable everywhere dense set. i.e. a countable set $M \subset X \ni [M] = T$

<u>proof:</u> let $\mathcal{G} = \{G_1, G_2, \dots\}$ be the countable base for $(X, \mathcal{T})$. For each G_i, choose $x_i \in G_i$. Define $M = \{x_i\}$. M is certainly countable. Claim: $[M] = X$. Assume not. then $X - [M] \neq \phi$. and $X - [M]$ is open. So we have a nonempty open set $\Rightarrow \exists$ some G_{n_0} such that $G_{n_0} \subset X - [M]$ because $\mathcal{G}$ is a base $\cancel{\cdot}$ because $x_{n_0} \in G_{n_0}$ and $x_{n_0} \notin X - [M]$.

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When does a countable everywhere dense subset guarantee a countable base? This is certainly true for metric spaces.

Theorem: consider (X, ρ) a metric space. Let (X, τ) be the topological space where the topology is generated by the metric ρ. If (X, ρ) has a countable everywhere dense subset then (X, τ) has a countable base.

proof:

Let $\{x_1, x_2, \dots\}$ be the countable everywhere dense subset. Then given an open set $G \subset X$ and any $x \in G$, $\exists$ sphere

$$S(x_m, 1/n) \subset G \text{ for some } x_m, n.$$

$\Rightarrow \{S(x_m, 1/n)\}$ is a countable base for the topology τ. //

So a separable metric space satisfies the second axiom of countability.

<u>Note:</u> $l^\infty(\mathbb{R}, \mathbb{N})$ has a countable local base but no countable base because it's not separable!

(Why is $l^\infty(\mathbb{R}, \mathbb{N})$ not separable? look at the following subset of $l^\infty(\mathbb{R}, \mathbb{N})$: take all sequences of 0's and 1's. there are uncountably many such sequences and each sequence is distance 1 away from each other sequence.
$\Rightarrow$ each one is in an l^∞ ~~ba~~ sphere of radius $1/2$ of its own. If $\exists$ a dense subset then it must have an element in each of these spheres $\Rightarrow$ it must be uncountable.)

<u>defn.</u> (X, τ) satisfies the first axiom of countability if each $x \in X$ has a countable local base $\mathcal{O}_x$ (i.e. if $G \in \tau$ and $x \in G$ then $\exists$ $O \in \mathcal{O}_x$ such that $x \in O$ and $O \subset G$.

so $l^\infty(\mathbb{R}, \mathbb{N})$ satisfies 1st axiom of countability, but not 2nd

Notes: if (X, τ) satisfies 2nd axiom of countability then it satisfies the first automatically.

defn: $x \in X$ is a contact point of a set $M \subset X$ if every neighborhood of X has $U \cap M \neq \emptyset$

defn: $x \in X$ is a limit point of M if every neighborhood of x has infinitely many points of M in it.

In a metric space, x is a contact point of M if and only if $\exists$ a sequence in M that converges to x.

In a topological space, x can be a contact point of M without M having a sequence that converges to x.

note: if $x \in M$ then x is a contact point and $\exists$ a sequence in M that converges to x

In (X,τ) a sequence $x_n \to x$ if for each neighborhood U of x, $\exists\, N_U$ such that $x_n \in U$ for all $n \geq N_U$

So far this is all a bunch of definitions. Let us consider the following topological space:

$X = \mathbb{R}$

$U \subset \mathbb{R}$ is open if $\mathbb{R} - U$ is finite or countable.

also, $\emptyset$ is open.

First of all, it's easy to check that this is a topology. So let's move on an look at its properties as a topological space.

<u>Fact</u> Very few sequences converge in this (X,τ).

Assume $x \to x$ Let

$U = (\mathbb{R} - \{x_n\}) \cup \{x\}$. Then U is open. $\Rightarrow \exists\, N_U \ni$ $\forall n \geq N_U$ we have $x_n \in U$. $\Rightarrow \forall\, n \geq N_U$ we have $x_n = x$. $\Rightarrow$ only "eventually stationary" sequences converge!

Also, the sequential definition of continuity says nothing about the function at hand. If $f : (X, \tau) \to (Y, \tilde{\tau})$ satisfies

$$x_n \to x \quad \Rightarrow f(x_n) \to f(x)$$

then this is no restriction on f since the sequence is eventually stationary.

Let $f : (X, \tau) \to (\mathbb{R}, \text{metric topology})$

$f(x) = x^2$. Then f is not continuous at $a \in \mathbb{R}$. Why? Let $U = (a^2 - \varepsilon, a^2 + \varepsilon)$.

Then U is open in the target space. If and $\nexists\, O$, open set in (X, τ), such that

$O \subset f^{-1}(U)$. Because O open $\Rightarrow$

$O = \mathbb{R} - (\text{finite or countable set})$

$\Rightarrow$ O is unbounded

$\Rightarrow$ O has arbitrarily large elements

$\Rightarrow f(O) \not\subset (a^2 - \varepsilon, a^2 + \varepsilon)$.

Contact points versus limit points.

Let $M = (0, 1]$.

Then 0 is a contact point of M because every open set about 0 intersects M. But 0 is not a limit point of M because $\nexists \{x_n\} \subset M$ with $x_n \to 0$.

Wierder still... (X, τ) isn't first countable. Since to be first countable, you have to have a countable base at each point.

Why? Assume (X, τ) is first countable. Let $x \in X$ and let $\{U_i\}$ be the local base.

Then $V = \bigcup_1^\infty (\mathbb{R} - U_i)$ is a countable set.

Take $y \notin V$. Define $\tilde{U} = \mathbb{R} - \{y\}$. Then $\tilde{U}$ is an open set containing x. And if one of the $U_{i_0} \subset \tilde{U}$ then $\mathbb{R} - \tilde{U} \subseteq \mathbb{R} - U_{i_0} \subset V$

$\Rightarrow y \in V$ $\times$!

Okay.. so (X,τ) isn't first countable. How much worse can things get?

<u>defn:</u> (X,τ) is a T_1 space if given $x \neq y$

$\exists\ G_x, G_y \in \tau$ such that

$x \notin G_y$ and $y \notin G_x$ (where G_x is an open neighborhood of x and y an open neighborhood of y.)

<u>defn:</u> (X,τ) is a T_2 space (or Hausdorff) if given $x \neq y$ $\exists\ G_x, G_y \ni$

$x \in G_x$, $y \in G_y$ and $G_x \cap G_y = \emptyset$

Our space (X, co-finite topology) is a T_1 space (just take $G_x = \mathbb{R}-\{y\}$ $G_y = \mathbb{R}-\{x\}$) but it isn't a Hausdorff space!

Is this just an artificial example?

No! It's like the Zariski topology.

Zariski topology on $\mathbb{R}$: $U \subseteq \mathbb{R}$ is open in the Zariski topology if $U = \phi$ or $\mathbb{R}$ or if there's a polynomial $P_U(x)$ such that $U = \{x \mid P_U(x) \neq 0\}$

Zariski topology on $\mathbb{R}^2$: Let $\{p_1, \ldots p_n\}$ be a finite set of polynomials on $\mathbb{R}^2$. Let the set $U = \{(x,y) \mid P_j(x,y) \neq 0 \text{ for } j = 1 \ldots n\}$ is open (by definition). (The # n and the polynomials are arbitrary) Furthermore, these are the only sets.

Note: Showing that the Zariski topology on $\mathbb{R}$ is a topology is easy. Showing that the topology on $\mathbb{R}^2$ really is a topology (unions of open sets, finite intersections of open sets..) involves fundamental theorems about the structure of polynomial rings.