Review the existence and uniqueness of solutions to

$$\frac{dy}{dx} = f(x,y)$$
$$y(x_0) = y_0$$

1) We needed $f : G \to \mathbb{R}$ where G is an open subset of $\mathbb{R}^2$, f continuous on G and f satisfies a Lipschitz condition in y on G:

$$|f(x,y) - f(x,\tilde{y})| \le M|y-\tilde{y}|$$

2) We let $\tilde{G} \subset G$ be closed and defined

$$K = \max_{\tilde{G}} |f(x,y)|$$

(max exists since $\tilde{G} \subseteq \mathbb{R}^2$ is closed & bounded and f is continuous)

3) We used K to define a closed and complete space:

$$X = \left\{ \phi : [x_0-\delta, x_0+\delta] \to \mathbb{R} \;\middle|\; \begin{array}{l} \phi \text{ is continuous} \\ \phi \text{ satisfies cone condition} \end{array} \right\}$$

cone condition:

$$|\phi(x) - y_0| \le K|x-x_0| \quad \forall x \in [x_0-\delta, x_0+\delta]$$

X is closed and complete <u>only</u> wrt L^∞ metric.
(to show X closed, just check that if $\phi_n \in X$
and $\phi_n \to \phi$ then $\phi \in X$.)

4) Having constructed X, we check that

$$A : X \to X.$$

i.e. if $\phi \in X$ then $A\phi \in X$. This is true precisely because of the cone condition.

5) Since A takes $X \to X$, we now choose δ small to ensure A is a <u>contraction</u>

$$\|A\phi - A\tilde{\phi}\| = \sup_{x \in [x_0-\delta, x_0+\delta]} |A\phi(x) - A\tilde{\phi}(x)|$$

$$= \sup \left| \int_{x_0}^{x} f(t, \phi(t)) - f(t, \tilde{\phi}(t))\, dt \right|$$

$$\leq \sup \int_{x_0}^{x} |f(t, \phi(t)) - f(t, \tilde{\phi}(t))|\, dt$$

$$\leq \sup \int_{x_0}^{x} M\, |\phi(t) - \tilde{\phi}(t)|\, dt$$

$$\leq \|\phi - \tilde{\phi}\| \sup \int_{x_0}^{x} M = M\delta\, \|\phi - \tilde{\phi}\|$$

take $\delta < 1/M$ and done!

Go back to the example of

$$\frac{dy}{dx} = y^2$$
$$y(0) = 1$$

Here, $f(x,y) = y^2$ (certainly continuous on $\mathbb{R}^2$)

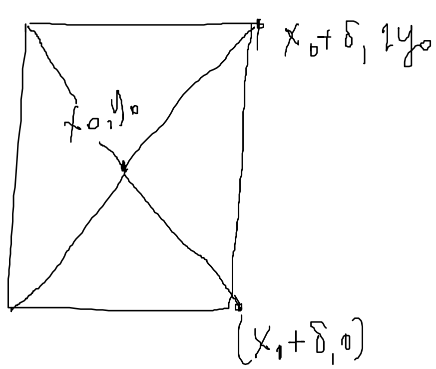
and f is Lipschitz in y if y is in a bounded set:

if $(x,y) \in \tilde{G}$

$$|f(x,y) - f(\tilde{y})| = |y^2 - \tilde{y}^2| = |y+\tilde{y}|\,|y-\tilde{y}|$$
$$\leq M|y-\tilde{y}|$$

as long as $|y+\tilde{y}| \leq M$ (need to stop $|y| \to \infty$.)

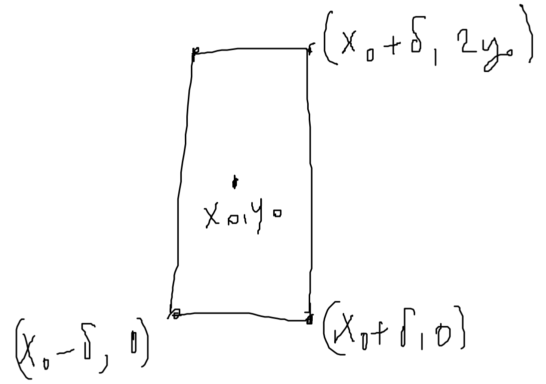
Let's get specific. We have initial data (x_0, y_0). I'll put $\tilde{G} =$



then $K = \max_G f(x,y)$

$$= \max_G y^2$$
$$= 4y_0^2$$

We have



the uniform bound on this $\tilde{G}$ is

$$\max_{\tilde{G}} |f(x,y)| = \max_{\tilde{G}} |y^2| = 4y_0^2 = K$$

and the Lipschitz constant

$$M = \max_{\tilde{G}} |y+\tilde{y}| = 4y_0$$

We now choose δ small enough for both the cone condition and the $M\delta<1$ condition to be satisfied.

Cone condition:

$$y_0 + \delta K \leq 2y_0$$

$$\delta K \leq y_0 \Rightarrow \delta \quad y_0/K = \frac{y_0}{4y_0^2} = \frac{1}{4y_0}$$

Contraction condition

$$\delta < \frac{1}{M} = \frac{1}{4y_0}$$

So we need $\delta \leq \lambda \frac{1}{y_0}$ for some $< \frac{1}{4}$

So we have a rule that works for the specific case of $f(x,y) = y^2$, when our $\tilde{G}$ is centered at (x_0, y_0) and has vertices $(x_0 \pm \delta, 0)$ and $(x_0 \pm \delta, 2y_0)$.

Now let's start at

$$(x_0, y_0) = (0, 1)$$

we have a solution at $x_0 + \delta_0 = \delta_0$ where $\delta_0 = \lambda \frac{1}{y_0} = \lambda$

We take $(\delta_0, y(\delta_0))$ as our new center point.

$y(\delta_0) = 1/1-\delta_0$ will determine δ_1

$$\delta_1 = \lambda \cdot \frac{1}{y(\delta_0)} = \lambda(1-\delta_0)$$

and we'll have a solution up to $\delta_0 + \delta_1 =$

$$\delta_0 + \lambda(1-\delta_0)$$
$$= \lambda + \delta_0(1-\lambda)$$
$$= \lambda + \lambda(1-\lambda)$$

We take $(\delta_0 + \delta_1, y(\delta_0 + \delta_1))$

$$= (\lambda + \lambda(1-\lambda), \ y(\lambda + \lambda(1-\lambda)))$$

as our new center point

$y(\delta_0 + \delta_1) = \dfrac{1}{1-(\delta_0+\delta_1)}$ will determine δ_2

Specifically,

$$\delta_2 = \lambda \frac{1}{y(\delta_0+\delta_1)} = \lambda\left(1-(\delta_0+\delta_1)\right)$$

$$= \lambda\left(1-[\lambda+\lambda(1-\lambda)]\right)$$

and we'll have our new center point

$$\left(\delta_0+\delta_1+\delta_2,\ y(\delta_0+\delta_1+\delta_2)\right)$$

where

$$\delta_0+\delta_1+\delta_2 = (\delta_0+\delta_1) + \lambda\left(1-(\delta_0+\delta_1)\right)$$

$$= \lambda + (1-\lambda)(\delta_0+\delta_1)$$

$$= \lambda + (1-\lambda)\left(\lambda+\lambda(1-\lambda)\right)$$

$$= \lambda + (1-\lambda)\lambda\left(1+(1-\lambda)\right)$$

Continuing in this way, after n steps our centerpoint is

$$\delta_0+\delta_1+\dots+\delta_n = \lambda + \lambda(1-\lambda)\left[1+(1-\lambda)+(1-\lambda)^2+\dots+(1-\lambda)^{n-1}\right]$$

$$= \lambda + \lambda(1-\lambda)\,\frac{1-(1-\lambda)^n}{1-(1-\lambda)}$$

$$= \lambda + (1-\lambda)\left[1-(1-\lambda)^n\right]$$

$$\lim_{n\to\infty} \delta_0+\delta_1+\dots+\delta_n = \lim_{n\to\infty} \lambda + (1-\lambda)\left[1-(1-\lambda)^n\right] = 1$$

So this shows that as you try to continue the solution to the right, you take smaller and smaller δ-steps at each step and

as $n \to \infty$ $\quad x_n \uparrow 1$

Our solution cannot be extended past $x=1$ since $y(x) \infty$ as $x_n \uparrow 1$.. this is reflected in the Lipschitz constant M blowing up as $x_n \uparrow 1$.

On the other hand, if you try to continue the solution from $x_0 - \delta_0 = -\lambda$ then you see that taking $\delta_1 = \lambda$ works $\Rightarrow$ continue to $x_0 - \delta_0 - \delta_1 = -2\lambda$ and $\delta_2 = \lambda$ works too $\Rightarrow$ continue to $x_0 - \delta_0 - \delta_1 - \delta_2 = -3\lambda$ and so on. In this way, $y(x)$ can be continue forever as $x \downarrow -\infty$.

Go back a bit... We started with $X = \{$continuous functions $| \ldots \}$ and we found $\phi \in X$ that satisfies the additional condition

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\,dt \quad \text{where } f(t,\phi(t)) \text{ is a cont. function of } t$$

(because $(t, \phi(t)) \in G$ where $f(x,y)$ is continuous.)

So the fundamental theorem of calculus then tells us, ϕ is differentiable!

so ϕ continuous, f continuous, and condition $A\phi = \phi$

$\Rightarrow$ ϕ differentiable!

Similarly, if we had been looking at X with the L^2 metric then we would have had to go to measure theory and would only know that the limit ϕ is a "measurable" function.

But

ϕ "measurable", ϕ satisfies $A\phi = \phi$,
& integrand $f(t, \phi(t))$ "measurable"

$\Rightarrow$ ϕ is differentiable at "almost all" points.

We don't really know what that means, except that we do know for sure that ϕ is much better than it would be if it didn't satisfy $A\phi = \phi$.

Please read the examples of solving integral equations and linear algebra problems using the contraction mapping theorem. They're important! (The ODE proof is the trickiest of them, though.)

Chapter 3 Topological Spaces

Metric spaces allowed us to talk about distances between points, distances between points and sets, distances between sets, diameters of sets.

So we had some idea of "small" sets from this.

What if there is no metric? We still have a lot of topology to help us out.

Basically, topology will do whatever we could do before using open + closed sets

Given a set X and a collection of subsets of X, τ, (X,τ) is a topological space if

1) $\phi \in \tau$
2) $X \in \tau$
3) $\{A_\alpha\} \in \tau \Rightarrow \bigcup_\alpha A_\alpha \in \tau$

(finite & countable & uncountable unions of elts of τ are in τ.)

4) $\bigcap_i^n A_i \in \tau$ if $A_i \in \tau$.

ex: if (X,ρ) is a metric space and τ = the set of open sets, then (X,τ) is a topological space.

ex: if τ = the set of all subsets of X then (X,τ) is a topological space (this is the space that corresponds to the metric

$$\rho(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

<u>ex:</u> if $\tau = \{X, \phi\}$ then (X, τ) is a t.s.

<u>ex:</u> if $X = \{a, b\}$ and

$\tau = \{a, b\}, \{b\}, \phi$

then (X, τ) is a topological space

<u>Note:</u> even in $\mathbb{R}$ w/ usual metric, the set of open sets is incredibly complicated.

the complement of the Cantor set is open. So it's $\in \tau$.

clearly, it'll be impossible to give all elts in the topology unless it's a really trivial topological space.

So we characterise the topology using "bases" if at all possible defn: A family $\mathcal{G}$ of opensets in (X,τ) is a base for (X,τ) if every open set in (X,τ) can be represented as a union of elts of $\mathcal{G}$

ex: In a metric space, take $\mathcal{G}$ = the set of all open spheres of all possible radii and centers

Why? Let $A \in \tau$. let $x \in A$. $\Rightarrow \exists\ r \ni$ $S(x,r) \subset A$ since A is open. let $A_x = A \cap S(x,r)$. then A_x is open.

$$\Rightarrow A = \bigcup A_x.$$

Well... that's a bit of a cheat, we're supposed to write A as a union of spheres. Really,

$A = \bigcup_x S(x,r)$. Why? Certainly $\bigcup_x S(x,r) \subseteq A$. And if $\bigcup_x S(x,r) \subsetneq A$ then $\exists$ $x_0 \in A - \bigcup_y S(x,r)$. But since A is open, $\exists\, r_0$ so that $S(x_0,r_0) \subset A$ and $S(x_0,r_0)$ is in the union. $\times$.

What we really need is an easy way to spot a base when we see one

<u>theorem</u>: Given a set X and let $\mathcal{G}$ be a subsystem of subsets of X that satisfies:

1) each $x \in X$ is in at least one $G_\alpha \in \mathcal{G}_\alpha$
2) if $x \in G_\alpha \cap G_\beta$ then $\exists\ G_\gamma \in \mathcal{G} \ni$
$$x \in G_\gamma \subset G_\alpha \cap G_\beta.$$

if we define ϕ and the set that contains all unions of elts of $\mathcal{G}$ as open, then (T, τ) is a topological space and and $\mathcal{G}$ is a base for (T, τ).

<u>proof</u>: First of all, ϕ and $X \in \tau$. and $\bigcup_\alpha$ open sets = open. All we need to do is show that finite intersections are in τ.

let $A, B \in \tau$. then

$$A = \bigcup_\alpha G_\alpha \quad \text{for some } \{G_\alpha\} \subset \mathcal{G}$$

$$B = \bigcup G_\beta \quad \text{" \quad " \quad "}$$

$$\Rightarrow A \cap B = \Big(\bigcup_\alpha G_\alpha\Big) \cap \Big(\bigcap_\beta G_\beta\Big)$$

$$= \bigcup_{\alpha,\beta} (G_\alpha \cap G_\beta)$$

now for each $x \in G_\alpha \cap G_\beta$ $\exists$ $G_\gamma \in \mathcal{G}$

with $x \in G_\gamma \subset G_\alpha \cap G_\beta$. $\Rightarrow$ $\bigcup_\gamma G_\gamma \subset \bigcup_{\alpha,\beta} (G_\alpha \cap G_\beta)$

and by same logic as before

$$\bigcup_\gamma G_\gamma = \bigcup_{\alpha,\beta} (G_\alpha \cap G_\beta)$$

$\Rightarrow$ $\bigcup_{\alpha,\beta} (G_\alpha \cap G_\beta) \in \tau$ and $A \cap B$ is open.

done! //

<u>theorem</u>: a system $\mathcal{G}$ of open sets G_α in a topological space (X, τ) is a base iff given any open set $G \in \tau$ and any $x \in G$, $\exists\, G_\alpha \in \mathcal{G} \ni$

$$x \in G_\alpha \subset G.$$

<u>proof</u>:

($\Longrightarrow$) if $\mathcal{G}$ is a base then every open set $G \in \tau$ is $G = \bigcup_\alpha G_\alpha$ for some collection of G_α's. $\Rightarrow$ every $x \in G$ is contained in some $G_\alpha \subset G$.

($\Longleftarrow$) Assume that given $G \in \tau$ $\exists\, G_\alpha(x) \in \mathcal{G}$ $\ni$ $x \in G_\alpha(x) \subset G$ for each $x \in G$ then

$$G = \bigcup_{x \in G} G_\alpha(x)$$

$\Rightarrow G$ is a union of elts of $\mathcal{G}$. //

<u>ex</u>: set of spheres w/ rational radii, centered at $x \in \mathbb{R}^n$ w/ rational coordinates.

<u>ex</u>: $\mathcal{G}$ = the set of open hyperplanes in $\mathbb{R}^n$.