Contraction Mapping Theorem

Every contraction mapping defined on a complete metric space has a unique fixed point.

A contraction mapping is a mapping

$$A : X \to X$$

$\ni$ $\rho(Ax, Ay) \le \alpha \rho(x,y)$ for some $\alpha < 1$
for all $x, y \in X$.

for example.

$A : \mathbb{R}^2 \to \mathbb{R}^2$

$A(x,y) = \left(\frac{x}{3}, y/4\right)$ is

$A(x,y) = \left(\frac{x}{3}, 2x\right)$ isn't, no matter what

$A(x,y) = \sqrt{x^2+y^2}\,(x,y)$ isn't. But is, if you restrict yourself to $S(\vec{0}, 1-\varepsilon)$. (the open ball of radius $1-\varepsilon$ centered at $(0,0)$)

$$A(x,y) = \begin{cases} \left(\frac{x}{3}, -2y\right) & \text{if } y > 0 \\ \left(\frac{x}{3}, -y/4\right) & \text{if } y < 0 \end{cases}$$

isn't, but A^2 is.

proof of CMT: We define a sequence $\{x_n\} \subset X$ and show it's Cauchy. Then completeness $\Rightarrow$ $x_n \to x$. We then show that $Ax = x$.

1) define sequence: $x_1 = Ax_0$, $x_2 = Ax_1$, $x_{n+1} = Ax_n$. the initial point x_0 is arbitrary.

2) Cauchy: fix m and n. Assume $m > n$

$$x_{n+1} = Ax_n$$
$$x_{n+2} = A^2 x_n$$
$$\vdots$$
$$x_m = A^{m-n} x_n$$

$$\begin{aligned}
\rho(x_m, x_n) &= \rho(A^{n+(m-n)} x_0, A^n x_0) \\
&\le \alpha^n \rho(A^{m-n} x_0, x_0) \\
&\le \alpha^n [\rho(x_0, Ax_0) + \rho(Ax_0, A^2 x_0) + \ldots + \rho(A^{m-n-1} x_0, A^{m-n} x \\
&\le \alpha^n [1 + \alpha + \alpha^2 + \ldots + \alpha^{m-n-1}] \rho(x_0, Ax_0) \\
&< \frac{\alpha^n}{1-\alpha} \rho(x_0, Ax_0) \qquad \underline{\text{since } \alpha < 1}
\end{aligned}$$

So given $\varepsilon > 0$ we choose $n \ni$ RHS $\frac{\alpha^n}{1-\alpha} |\rho(x_0, A$
$< \varepsilon$

this shows the sequence is Cauchy.
$\Rightarrow$ It converges to $x \in X$. Now show $Ax = x$.

$$Ax = A \lim_{n\to\infty} x_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_{n+1} = x$$

used continuity of A. (all contraction mappings are cts automatically.) //

Contraction Mapping Theorem Quite Useful
K+F gives linear algebra iterations existence of solutions to ODEs, existence of solutions to integral equations.

<u>ODEs:</u> Consider

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

Formally, $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$
is a solution, but how's that useful?

Define $A : X \to X$ by

$$Aw = y_0 + \int_{x_0}^{x} f(t, w(t))\,dt$$

certainly, if $Aw = w$ then we've found

$$w(x) = y_0 + \int_{x_0}^{x} f(t, w(t))\,dt$$

which is our desired solution.

When does this work?

ex:

$$\frac{dy}{dx} = f(x, y) = y$$

$$y(x_0) = 1 \qquad x_0 = 0$$

We need a first guess. The function $\equiv 1$ is a good start, one hopes

$$\phi_0(x) = 1$$

$$\phi_1(x) = 1 + \int_0^x f(t, \phi_0(t))\,dt$$

$$= 1 + \int_0^x \phi_0(t)\,dt = 1 + x$$

$$\phi_2(x) = \int 1 + \int_0^x \phi_1(t)\,dt = 1 + x + \frac{x^2}{2}$$

$$\phi_3(x) = 1 + \int_0^x \phi_2(t)\,dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$$

$\vdots$

$$\phi_n(x) = \sum_0^n \frac{x^j}{j!}$$

clearly
$$\phi_n \to \sum_0^\infty \frac{x^j}{j!} = e^x \qquad \forall x \in \mathbb{R}.$$
So this construction gives us a solution on an infinite interval containing our initial point x_0.

ex:
$$\frac{dy}{dx} = f(x,y) = y^2$$
$$y(0) = 1$$

gives
$$\phi_0(x) = 1$$
$$\phi_1(x) = 1 + x$$
$$\phi_2(x) = 1 + x + x^2 + \frac{x^3}{3}$$
$$\phi_3(x) = 1 + x + x^2 + x^3 + \frac{2}{3}x^4 + \dots + \frac{1}{63}x^7$$
$$\phi_4(x) = 1 + x + x^2 + x^3 + x^4 + \frac{13}{15}x^5 + \dots + \frac{1}{59535}x^{15}$$

presumably
$$\phi_n \to \sum_0^\infty x^j \qquad \text{as } n \to \infty$$
$$= \frac{1}{1-x} \qquad \text{if } |x| < 1$$
This example suggests a solution that exists on a finite interval that contains

$x_0 = 0$. In reality, we know the solution $y(x) = \frac{1}{1-x}$

and this does just fine for $x \in (-\infty, 1)$

So the power series argument's missing the $x \leq -1$.

Okay, let's <u>prove</u> the existence of solutions (unique) to the ODE $\frac{dy}{dx} = f(x,y)$ $y(x_0) = y_0$.

And we'll see what happens...

<u>defn</u>: $f(x,y)$ is Lipschitz in y in a domain G

if $|f(x,y) - f(x,\tilde{y})| \leq M|y - \tilde{y}|$

for some M, for all $y \in G$

<u>theorem</u> (Picard) Given $f(x,y)$ defined and continuous in a plane domain $G \ni (x_0, y_0)$, and assume f is Lipschitz in y in this domain. Then $\exists$ an interval $|x - x_0| \leq \delta$ in which the ODE

$$\frac{dy}{dx} = f(x,y)$$

has a unique solution satisfying $y(x_0) = x_0$

the interval $[x_0-\delta, x_0+\delta]$ is an interval of existence. Q! is it the largest interval of existence? what happens if you take your solution and define $y_1 = y(x_0+\delta)$

and construct another solution w/ initial data

$$y(x_0+\delta) = y_1 \text{ ?}$$

will this extend your solution or will the construction break down for some reason?

proof: We want to find $\phi(x)$ that satisfies

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt$$

Let $G' \subset G$ be a closed subset of G

Since f is cts on G', we know

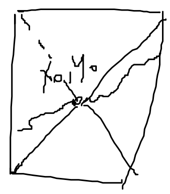
$$|f(x,y)| \leq K$$

for some K, for all $(x,y) \in G'$.

Let $\delta > 0$ and define

$$X = \text{continuous functions on } [x_0-\delta, x_0+\delta].$$

Okay, we've got a plausible set,



our trajectory should live in X, if we choose δ right.

Note: $L^\infty([x_0-\delta, x_0+\delta])$ is complete --- it's the only L^p space that is complete (if we only consider continuous functions.) And X is a closed subset of $L^\infty([x_0-\delta, x_0+\delta])$.

So we have two things to show:

1) $A : X \to X$
2) A is a contraction on X.

Hopefully, if we choose δ small enough, we can guarantee both of these. And then the contraction mapping theorem will give us a fixed point of A. Which is a solution of the ODE by construction..

First, let's see if $A: X \to X$.

$$\psi = A\phi = y_0 + \int_{x_0}^{x} f(t, \phi(t))\,dt.$$

Assume $\phi \in X$, show $\psi \in X$.

$$|\psi(x) - y_0| = \left|\int_{x_0}^{x} f(t, \phi(t))\,dt\right|$$

$$\leq \int_{x_0}^{x} |f(t, \phi(t))|\,dt \leq M|x - x_0|$$

$$\leq M\delta \quad ✓$$

So A takes $X \to X$. ✓

Now show A is a contraction.

$$\|A\phi - A\tilde{\phi}\| = \sup_{x \in [x_0 - \delta,\, x_0 + \delta]} \left|\int_{x_0}^{x} f(t, \phi(t)) - f(t, \tilde{\phi}(t))\,dt\right|$$

$$\leq \sup \int_{x_0}^{x} |f(t, \phi(t)) - f(t, \tilde{\phi}(t))|\,dt$$

$$\leq \sup \int_{x_0}^{x} M|\phi(t) - \tilde{\phi}(t)|\,dt$$

$$\leq M\|\phi - \tilde{\phi}\| \sup \int_{x_0}^{x} dt$$

$$\leq M\|\phi - \tilde{\phi}\|\,\delta.$$

So if we choose $\delta \ni M\delta < 1$ then we have a contraction!

that's our definition of "sufficiently small" in this case

So if we choose $\delta < \frac{1}{M}$ then we have a unique solution of the ODE on $[x_0 - \delta, x_0 + \delta]$ //

<u>Note</u>: The above is the G-rated version of an application of the contraction mapping theorem.

A more standard result would be like:

1) prove $\exists$ a solution in L^2
2) notice that L^2 functions aren't that nice (they aren't necc continuous or differentiable.)
3) Put more work in and show that even though $x_n \to x$ where the limit x is in a weak space it's really better than you think. (For examples for the ODE argument, $y(x)$ is differentiable precisely because it satisfies $y(x) = y_0 + \int_{x_0}^{x}$ somethin

Kolmogorov + Fomin use the contraction mapping theorem to prove existence of solutions in L^∞. Here, we're lucky in that L^∞ (when defined in terms of continuous functions, which is what we're doing until we do measure theory) is complete.

One of your HW problems is to prove the existence of a limit with respect to the L^2 metric. In this case, you

1) prove $x_n \to x$
2) prove x is actually continuous, even though arbitrary limit points of continuous functions in L^2 aren't continuous.