You know what a Cauchy sequence is:
given $\varepsilon > 0$ $\exists N \ni m, n \geq N$
$\Rightarrow \rho(x_n, x_m) < \varepsilon$

(K+F calls them fundamental sequences)

<u>theorem</u>: If $\{x_n\}$ is convergent then it is Cauchy

<u>proof</u>: See K+F.

<u>note</u>: "convergent" means $\exists\ x \in X \ni x_n \to x$. Cauchy doesn't imply that. (Clear since you can have a Cauchy sequence in $\mathbb{Q}$ whose limit is irrational.)

A metric space is <u>complete</u> if every Cauchy sequence in X converges to an elt of X.

<u>theorem:</u> $l^p(\mathbb{R}, \mathbb{N})$ is complete for $1 \le p \le \infty$

<u>proof:</u> let $\{x_n\}$ be cauchy in $l^p(\mathbb{R}, \mathbb{N})$.

Given $\varepsilon > 0$ $\exists$ $N \ni$ $m, n \geq \mathbb{N} \Rightarrow$

$$\rho(x_n, x_m) < \varepsilon$$

i.e. $$\sqrt[p]{\sum_1^\infty |(x_n)_i - (x_m)_i|^p} < \varepsilon$$

$\left(\text{or } \sup_i |(x_n)_i - (x_m)_i| < \varepsilon \quad \text{if } p = \infty\right)$

$\Rightarrow$ for each i, we have

$$|(x_n)_i - (x_m)_i| < \varepsilon \quad \text{if } m, n \geq N$$

$\Rightarrow$ for each i, $\{(x_n)_i\}$ is a Cauchy sequence in $\mathbb{R}$. Well $\mathbb{R}$ is complete

$\Rightarrow$ $\exists$ $x_i \in \mathbb{R}$ $\ni$ $(x_n)_i \to x_i$ as $n \to \infty$

Let $x = \{x_i\}$. So we've found our candidate for a limit. We now have to prove 1) $x_n \to x$ and 2) $x \in \ell^p(\mathbb{R}, \mathbb{N})$.

First, show $x \in \ell^p$. Fix $\varepsilon > 0$. Then $\exists N \ni m, n \geq N \Rightarrow$

$$S_M = \sum_1^M |(x_m)_i - (x_n)_i|^p < \varepsilon$$

Hold n fixed and let $m \to \infty$. Then

$$\tilde{S}_M = \sum_1^M |x_i - (x_n)_i|^p \leq \varepsilon$$

$\Rightarrow$ we have a uniform upper bound on $\tilde{S}_M$. So we can take $M \to \infty$ and we've shown that for this fixed n,

$$\sum_1^\infty |x_i - (x_n)_i|^p \leq \varepsilon$$

$\Rightarrow$ $x - x_n \in \ell^p$ and $x_n \in \ell^p$

$\Rightarrow$ $x \in \ell^p$ (check this!)

So $x \in \ell^p(\mathbb{R}, \mathbb{N})$. Also, we proved $x_n \to x$ along the way because we showed that for $n \geq N$ we have

$$\rho(x_n, x) \leq \varepsilon_n$$

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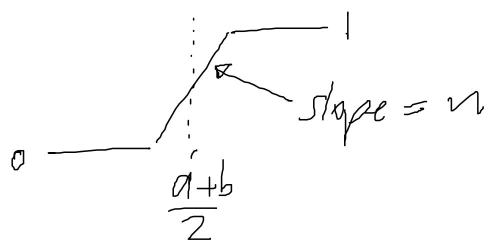
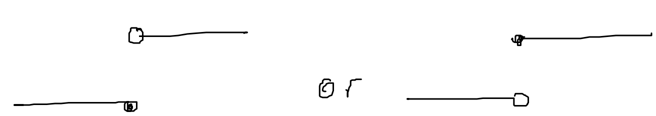
<u>Ex:</u> if I define $L^p([a,b])$ to be the space of functions that are continuous on $[a,b]$ and have

$$\rho(f,g) = \sqrt[p]{\int_a^b |f(x)-g(x)|^p \, dx}$$

then (X, ρ) is <u>not</u> complete.

<u>proof:</u> let

$f_n =$ 

then $\lim_{n \to \infty} f_n =$ 

Eek! Not only is it not complete, the limit doesn't even look unique.

$S[x_1, r_1]$ = closed sphere centered at x_1

$= \{y \in X \mid \rho(x, y) \le r_1\}$

$S[x_1,r_1] \supseteq S[x_2,r_2] \supseteq S[x_3,r_3] \supseteq \ldots$

is a nested sequence of closed spheres.

<u>nested sphere theorem</u>: A metric space (X, ρ) is complete iff every nested sequence of spheres $\ni r_n \to 0$ satisfies

$$\bigcap_1^{\infty} S[x_n, r_n] \neq \emptyset$$

<u>proof</u>:

($\Rightarrow$) Assume (X, ρ) is complete.

Since $r_n \to 0$ we know that $\{x_n\}$ is a cauchy sequence. (Given $\varepsilon > 0$ choose $N \ni n \ge N \Rightarrow r_n < \varepsilon$. Then

$\forall n \ge N \quad S_n[x_n, r_n] \subseteq S_N[x_N, r_N]$

$\Rightarrow \rho(x_n, x_m) < \varepsilon$ for $m, n \ge N$

Since X is complete, we know limit $\exists$. $x \in X$. Claim that $x \in \bigcap_1^\infty S[x_n, r_n]$.

Why? Fix some $S_N[x_n, r_n]$. It contains all but finitely many of the x_n. $\Rightarrow$ x is a limit point of $S[x_n, r_n]$.

Since $S[x_n, r_n]$ is closed we know $x \in S[x_n, r_n]$. The index n was arbitrary, so $x \in S[x_n, r_n]$ $\forall n \Rightarrow x \in \bigcap_1^\infty S[x_n, r_n]$.

($\Leftarrow$) Assume every nested sequence of spheres has a nonempty intersection. Let $\{x_n\} \subset X$ be Cauchy. We use the sequence to construct a nested closed family.

For $1/2$ $\exists$ N_1 $\ni$ $n, m \geq N_1 \Rightarrow$

$$\rho(x_n, x_m) < 1/2.$$

specificall, $\rho(x_{N_1}, x_m) < 1/2$ $\forall m \geq N_1$

For $\frac{1}{2^2}$ $\exists N_2 \ni m, n \geq N_2$

$\Rightarrow \rho(x_n, x_m) < 2^{-2}$

specifically, $\rho(x_{N_2}, x_n) < 2^{-2}$ $\forall n \geq N_2$

And so on, construct a subsequence

$x_{N_1}, x_{N_2}, x_{N_3}, \ldots \ni$

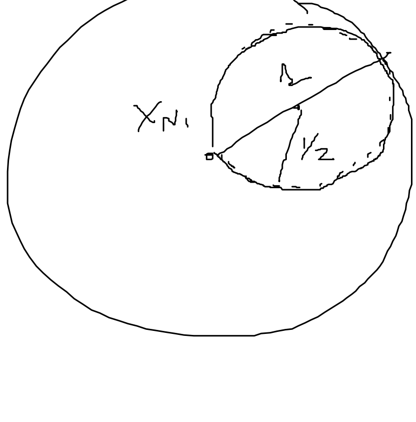
if $N_k \geq N_l$ then

$$\rho(x_{N_k}, x_{N_l}) < 2^{-l-1}$$

We've got our centers, now we define our spheres.

$S[x_{N_1}, r_1] =$ sphere of radius 1

then sphere of radius $\frac{1}{2}$ centered about x_{N_2} will be in $S[x_{N_1}, r_1]$:



and sphere of radius $\frac{1}{4}$ centered about x_{N_3} will be in $S[x_{N_2}, r_2]$ and so on

We have our nested family and

let $x \in \bigcap_{1}^{\infty} S[x_n, r_n]$

and $x_{N_k} \to x$ by construction. It follows that $x_n \to x$ because if a subsequence of a Cauchy sequence converges to x then the sequence itself converges to x. //

Why did we need closed spheres? Let's look at

$$S(1/2n, 1/2n) = (0, 1/n) \quad \text{in } \mathbb{R}$$

then they're nested, but their intersection is empty.

Baire theorem: A complete metric space (X, ρ) cannot be represented as the union of a countable number of nowhere dense sets.

Corollary: $\mathbb{Q}$ is not complete

proof of B.T. Assume not. Assume

$$X = \bigcup_1^\infty A_n \quad \text{where each } A$$

is nowhere dense. Fix A_1. Since A_1 is nowhere dense, given any sphere (open) in X, $[A_1 \cap S] \neq S \Rightarrow S - [A_1 \cap S]$ is open $\Rightarrow \exists\ \tilde{S}$ open $\ni \tilde{S} \subset S - [A_1 \cap S]$ i.e. $\tilde{S} \cap A_1 = \phi$. take $\tilde{\tilde{S}}$ closed in $\tilde{S}$ then $\tilde{\tilde{S}} \cap A_1 = \phi$. This is true for any sphere S, so let's start w/ S = a sphere of radius 1. $\Rightarrow \exists\ \tilde{\tilde{S}}$ a closed sphere of radius $\leq \frac{1}{2} \subset S \ni \tilde{\tilde{S}} \cap A_1 = \phi$

Now, A_2 is nowhere dense too $\Rightarrow \exists$ a sphere of radius $\leq 1/4$ living in S_2 such that $S_3 \subset S_2$ and $S_3 \cap A_2 = \emptyset$ (notice $S_3 \cap A_1 = \emptyset$ already!) Keep going, and you construct a nested sequence of closed spheres

$$S_2 \supseteq S_3 \supseteq S_4 \supseteq \cdots$$

$$\ni \quad S_2 \cap A_1 = \emptyset$$

$$S_3 \cap A_1 = S_3 \cap A_2 = \emptyset$$

$$S_n \cap A_i = \emptyset \qquad i \in \{1, \ldots n-1\}$$

by the nested sphere theorem $\bigcap_1^\infty S_n \neq \emptyset$

$\Rightarrow x \in \bigcap_1^\infty S_n \Rightarrow x \in X$ but $x \notin A_i \ \forall i$

$\Rightarrow \bigcup_1^\infty A_i \neq X$. //